A study of particular methods for the approximate construction of some regular polygons by using Mathematica 3.0

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Abstract

In this paper we study particular methods for the approximate construction of the regular pentagon, heptagon, enneagon, decagon and undecagon by using *Mathematica 3.0*. In the same way, we make a study of the errors of those methods whose construction is not exact, and we compare them with those obtained in the general methods of Archimedes and Bardin. We also include the coded algorithm in *Mathematica 3.0* for the animation of each method.

1 Preliminaries

Let us LE_n denote the exact longitude on the side of the regular polygon of n sides inscribed in a circumference of radius 1, this is

$$LE_n = 2 \text{ sen} \left(\frac{\pi}{n}\right) \quad \forall n \geq 3,$$

and L_n the longitude on the side of the regular polygon of n sides inscribed in the same circumference constructed by the corresponding particular algorithm.

In the other hand, let $d(P_1, P_2)$ be the euclidean distance between the points $P_1, P_2 \in \mathbb{R}^2$, and $C[P, r] = C[(a, b), r]$ the circumference centered on $P = (a, b)$ and radius r. In particular $C[(0, 0), 1]$ is the goniometric circumference and $C[P, d(M, N)]$ the circumference centered on P and radius equal to the distance between M and N.

2 Construction of the regular pentagon

2.1 Description of the algorithm

1) Trace the circumference $C[(0, 0), 1]$ and its principal diameters AB and CD (as main diameters we understand the two perpendicular diameters that coincide with the coordinated axes).

2) Determinate the point P_1 of negative ordinate, which is result of the intersection of $C[(0, 0), 1]$ and $C[(1, 0), 1]$.

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3) Determinate the point P_2 of negative abscissa, which is result of the intersection of $C[(0,0),1]$ and $C[(0,-1),1]$.

4) Determinate the point P_3 of negative abscissa, which is result of the intersection of $C[P_1, d(P_1,P_2)]$ and the diameter CD.

5) The distance $L_5 = d(B, P_3)$ is the side of the regular pentagon that we want to draw.

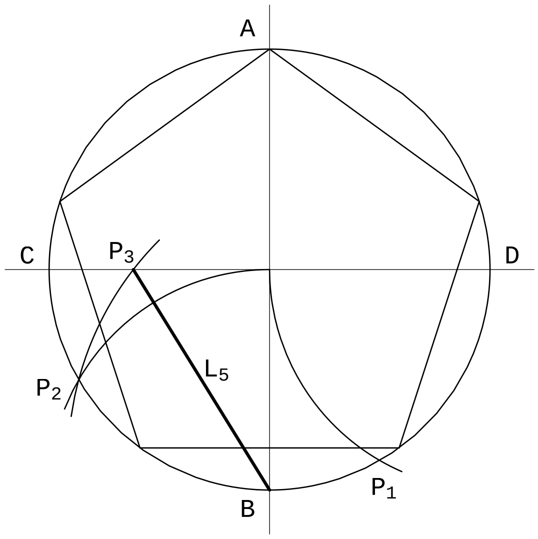

Figure 1: Construction of the pentagon

2.2 Analysis of the error

Let us prove, by using *Mathematica 3.0*, that above algorithm is exact, this is, that $L_5 = LE_5$.

The points P_1 and P_2 can be founded by means of

$$\mathbf{Solve[\{x^2+y^2 == 1, (x-1)^2+y^2 == 1\}, \{x,y\}]}$$
$$\mathbf{Solve[\{x^2+(y+1)^2 == 1, x^2+y^2 == 1\}, \{x,y\}]}$$

resulting

$$P_1 = \left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right) \quad \text{and} \quad P_2 = \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right).$$

To calculate P_3 we previously need compute the distance between P_1 and P_2, that it can be obtain by using

$$\mathbf{distance[\{a\_, b\_\}, \{c\_, d\_\}] := \sqrt{(c-a)^2+(d-b)^2}}$$
$$\mathbf{dis(P1,P2) := distance[\{1/2, -\sqrt{3}/2\}, \{-\sqrt{3}/2, -1/2\}]}$$
$$\mathbf{Solve[\{(x-1/2)^2+(y+\sqrt{3}/2)^2 == dis(P1,P2)^2, y == 0\}, \{x,y\}]}$$

outputting

$$P_3 = \left(\frac{1-\sqrt{5}}{2}, 0\right).$$

Finally, the side L_5 can be calculated in the way

$$\mathbf{Lado5 := distance[\{0, -1\}, \{(1 - \sqrt{5})/2, 0\}]}$$

that is equal to LE_5 as can be proved by means of the following algorithm

$$\mathbf{ExactSide[n\_] := 2 * Sin\left[\frac{\pi}{n}\right]}$$
$$\mathbf{Abs[ExactSide[5] - Lado5]}$$

3 Construction of the regular heptagon

3.1 Description of the algorithm

1) Trace the circumference $C[(0,0),1]$ and its principal diameters AB and CD.
2) Determinate the point P_1 and P_2, result of the intersection of $C[(0,0),1]$ and $C[(0,-1),1]$.
3) Determinate the point P_3, result of the intersection of the straight line joining P_1 with P_2 and the diameter AB.
4) The distance $L_7 = d(P_1, P_3)$ is the side of the regular heptagon that we want to draw.

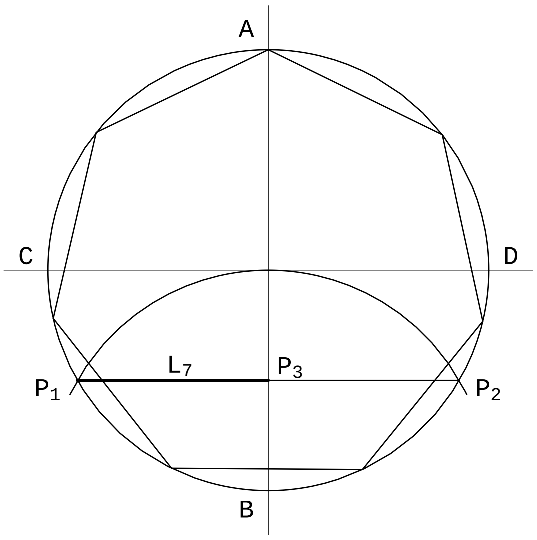

Figure 2: Construction of the heptagon

3.2 Analysis of the error

Let us start determinating the points P_1 and P_2

$$\mathbf{Solve[\{x^2 + y^2 == 1, x^2 + (y+1)^2 == 1\}, \{x, y\}]}$$

obtaining

$$P_1 = \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right) \quad \text{y} \quad P_2 = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right),$$

and therefore

$$P_3 = \left(0, \frac{-1}{2}\right).$$

Hence, we can calculate L_7 by means of

$$\mathbf{Lado7} := \mathbf{distance}[\{\mathbf{0}, -\mathbf{1/2}\}, \{(-\sqrt{\mathbf{3}}/\mathbf{2}, -\mathbf{1/2}\}]$$

resulting the value $L_7 = \sqrt{3}/2$.

We will obtain an estimate of the precision of the method starting from the *relative error* $(LE_7 - L_7)/LE_7$. The reason why we don't consider the error taking the absolute value of the difference $(LE_7 - L_7)$, is that we want to know if the value L_7 obtained by means of the algorithm approaches for excess or for defect to the exact longitude LE_7. In particular, if the relative error is positive then the approach is for defect, else it is for excess.

The code

$$\mathbf{RelativeErrorHeptagon} := (\mathbf{ExactSide}[\mathbf{7}] - \mathbf{Lado7})/\mathbf{ExactSide}[\mathbf{7}]$$

produces the value $2.00754 \cdot 10^{-3}$, i.e., the error of the method is roughly 0.2%.

4 Construction of the regular enneagon

4.1 Description of the algorithm

1) Trace the circumference $C[(0,0),1]$ and its principal diameters AB and CD.

2) Determinate the point P_1 of positive abscissa , which is result of the intersection of $C[(0,0),1]$ and $C[(0,-1),1]$.

3) Determinate the point P_2 of positive abscissa , which is result of the intersection of $C[(0,1),d(A,P_1)]$ and the diameter CD.

4) Determinate the point P_3 of negative abscissa , which is result of the intersection of $C[P_2,d(P_2,B)]$ and the diameter CD.

5) The distance $L_9 = d(C,P_3)$ is the side of the regular enneagon that we want to draw.

4.2 Analysis of the error

Let us calculate P_1 by means of the code

$$\mathbf{Solve}[\{\mathbf{x^2} + \mathbf{y^2} == \mathbf{1}, \mathbf{x^2} + (\mathbf{y}+\mathbf{1})^{\mathbf{2}} == \mathbf{1}\}, \{\mathbf{x}, \mathbf{y}\}]$$

resulting

$$P_1 = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right).$$

To determinate P_2 can be used the following algorithm

$$\mathbf{dis(A,P1)} := \mathbf{distancia}[\{\mathbf{0}, \mathbf{1}\}, \{\sqrt{\mathbf{3}}/\mathbf{2}, -\mathbf{1/2}\}]$$
$$\mathbf{Solve}[\{\mathbf{x^2} + (\mathbf{y}-\mathbf{1})^{\mathbf{2}} == \mathbf{dis(A,P1)^2}, \mathbf{y} == \mathbf{0}\}, \{\mathbf{x}, \mathbf{y}\}]$$

which outputs $P_2 = \{\sqrt{2}, 0\}$.

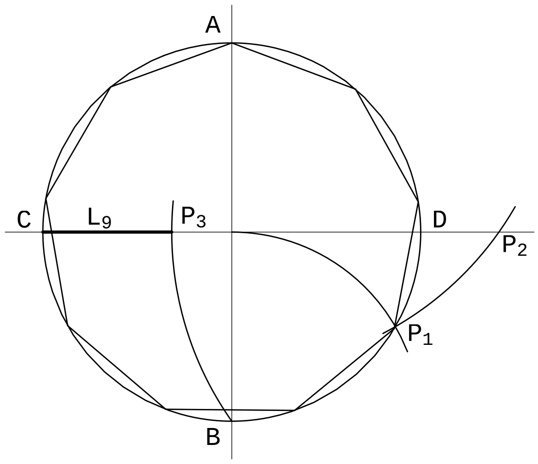

Figure 3: Construction of the enneagon

On the other hand, we can find P_3 in the way

$$\mathbf{dis(P2, B) := distance[\{\sqrt{2}, 0\}, \{0, -1\}]}$$
$$\mathbf{Solve[\{(x - \sqrt{2})^2 + y^2 == dis(P2, B)^2, y == 0\}, \{x, y\}]}$$

obtaining $P_3 = \{\sqrt{2} - \sqrt{3}, 0\}$.

Finally, the code

$$\mathbf{Lado9 := distance[\{-1, 0\}, \{\sqrt{2} - \sqrt{3}, 0\}]}$$

allows us to calculate $L_9 = 1 - \sqrt{3} + \sqrt{2}$.

In this case the relative error of the method is the output generated by the input

$$\mathbf{RelativeErrorEnneagon := (ExactSide[9] - Lado9)/ExactSide[9]}$$

resulting the value $2.74477 \cdot 10^{-3}$, i.e, the error of the method is roughly 0.3%.

5 Construction of the regular decagon

5.1 Description of the algorithm

1) Trace the circumference $C[(0,0),1]$ and its principal diameters AB and CD.
2) Determinate the point P_1 and P_2, result of the intersection of $C[(0,0),1]$ and $C[(0,-1),1]$.
3) Determinate the point P_3, result of the intersection of the straight line joining P_1 with P_2 and the diameter AB.
4) Determinate the point P_4 of negative abscissa, result of the intersection of the straight line joining P_3 with C and $C[P_3, d(P_3, B)]$
5) The distance $L_{10} = d(P_4, C)$ is the side of the regular decagon that we want to draw.

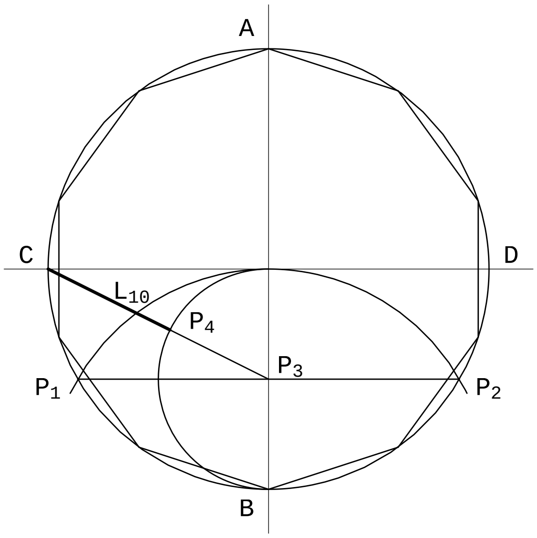

Figure 4: Construction of the decagon

5.2 Analysis of the error

We will check that the above algorithm allows us to construct the regular decagon in an exact way.

Observe that the points P_1, P_2 and P_3 coincide with its homologous ones in the case of the construction of the heptagon. To calculate P_4 we can use

$$\mathbf{Solve[\{x^2 + (y + 1/2)^2 == 1/4, y == -(1/2) * (x + 1)\}, \{x, y\}]}$$

obtaining

$$P_4 = \left(-\frac{\sqrt{5}}{5}, \frac{1}{10}(-5 + \sqrt{5})\right).$$

Therefore, L_{10} can be determinated by means of

$$\mathbf{Lado10 := distance[\{-1, 0\}, \left\{-\frac{\sqrt{5}}{5}, \frac{1}{10}(-5 + \sqrt{5})\right\}]}$$

resulting the value $L_{10} = (-1 + \sqrt{5})/2$.

To prove that this algorithm is exact, it is enough to check that the following value vanishes

$$\mathbf{Abs[ExactSide[10] - Lado10]}$$

6 Construction of the regular undecagon

6.1 Description of the algorithm

1) Trace the circumference $C[(0, 0), 1]$ and its principal diameters AB and CD.

2) Determinate the point P_1 of positive abscissa , which is result of the intersection of $C[(0, 0), 1]$ and $C[(0, -1), 1]$.

3) Determinate the point P_2, result of the intersection of the straight line joining P_1 with A and the diameter CD.

4) The distance $L_{11} = d(P_1, P_2)$ is the side of the regular undecagon that we want to draw.

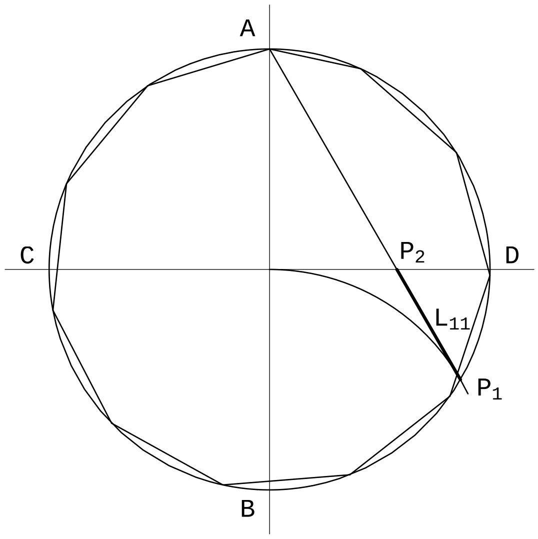

Figure 5: Construction of the undecagon

6.2 Analysis of the error

The point P_1 coincide with the point P_2 in the subsection 3.2, this is, $P_1 = (\sqrt{3}/2, -1/2)$.

Now, P_2 can be calculating by using

$$\mathbf{Solve}[\{\mathbf{y} == -\sqrt{\mathbf{3}}\mathbf{x} + \mathbf{1}, \mathbf{y} == \mathbf{0}\}, \{\mathbf{x}, \mathbf{y}\}]$$

obtaining $P_2 = (\sqrt{3}/3, 0)$.

Finally we can obtain L_{11} by means of the code

$$\mathbf{Lado11} := \mathbf{distance}[\{\sqrt{\mathbf{3}}/\mathbf{3}, \mathbf{0}\}, \{\sqrt{\mathbf{3}}/\mathbf{2}, -\mathbf{1}/\mathbf{2}\}]$$

which provides the value

$$L_{11} = \sqrt{\frac{1}{4} + \left(\frac{-\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\right)^2}.$$

Therefore, the relative error of the method

$$\mathbf{ErrorRelativoUndecagono} := (\mathbf{LadoExacto}[\mathbf{11}] - \mathbf{Lado11})/\mathbf{LadoExacto}[\mathbf{11}]$$

results $-2.46424 \cdot 10^{-2}$, i.e., in this occasion we approach for excess and the error is greater than the error of before cases.

7 Comparison with the general methods of Archimedes and of Bardin

The table 1 compare the relative errors of the particular algorithms for the construction of the pentagon, enneagon and undecagon with the relative errors that result of the layout of these polygons using the general methods of Archimedes and of Bardin.

n	Rel. Error of the particular algorithm	Rel. error Archimedes	Rel. Error Bardin
7	$2.00754 \cdot 10^{-3}$	$-1.62426 \cdot 10^{-3}$	$2.86033 \cdot 10^{-4}$
9	$2.74477 \cdot 10^{-3}$	$-6.65798 \cdot 10^{-3}$	$1.45450 \cdot 10^{-3}$
11	$-2.46424 \cdot 10^{-2}$	$-1.24955 \cdot 10^{-2}$	$1.78029 \cdot 10^{-3}$

Table 1: Comparison of the relative errors

It is curious that a general method as the Bardin's one get better that the corresponding particular methods. However, the particular construction of the enneagon is better than that obtained by the method of Archimedes.

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