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A Congruence relation in finite sPBC *

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Abstract

In this paper we define a congruence relation for regular terms of finite sPBC, by means of which we may identify those processes that have the same behaviour, not only in terms of the multiactions that they can perform, but also for the stochastic information that they have associated. In order to define this equivalence relation we need to consider an adequate semantics for the synchronization operator, as well as a new labelled transition system for regular terms of finite sPBC.

Keywords: Stochastic Process Algebras, Petri Box Calculus, Stochastic Petri Nets.

1 Introduction

Petri Box Calculus (PBC) is a process algebra where the real parallelism of concurrent systems can be naturally expressed [3, 5, 6, 7, 8, 10]. The main property of PBC is that its operators are selected in such a way that it is relatively easy to define a denotational semantics based on a class of labelled Petri net, called *boxes*. With this purpose synchronizations are separated from the parallel operator, by including a new operator (sy), in contrast to the way in which the synchronization is handled in classical process algebras, such as CCS [14], where the synchronization is embedded in the parallel operator.

PBC can be extended by including time or probabilistic information, with the goal to describe a wider class of systems, such as real-time systems and fault-tolerance systems. We may find in the literature two timed extensions of PBC, namely tPBC [9] and TPBC [13], which consider a deterministic model of time. In a previous paper [12] we presented a first version of sPBC, which is a stochastic extension of PBC.

In sPBC we consider that multiactions have associated Markovian stochastic delays, which are interpreted as the time that must elapse until the corresponding multiaction can be executed (computed from the instant in which it was activated), but the execution of multiactions

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takes no time. Therefore, in sPBC each multiaction has associated a random delay, which follows a negative exponential distribution. Thus, a (basic) stochastic multiaction is represented by a pair $<\alpha, r>$, where α represents a (classical) multiaction of PBC and $r \in \mathbb{R}^+$ is the parameter of the associated exponential distribution. Moreover, in case of conflict we adopt the *race policy*, i.e., whenever two or more stochastic multiactions are possible, the fastest one is executed, in the same way as this *race policy* governs the dynamic behaviour of stochastic Petri nets [1].

The denotational semantics of sPBC is defined using as semantic objects a special class of stochastic Petri net, which are called *s-boxes*. Note the main consequence of the introduction of a continuous distribution (negative exponential) to measure delays of multiactions: the probability for two or more stochastic multiactions to be executed at the same time is zero; and the same is true for stochastic Petri nets: two or more transitions have zero probability to be executed in parallel (even if they are simultaneously enabled). As a consequence, we get a total order semantics, i.e., a semantics that satisfies the *Total Order Assumption (TOA)* [2]: *all execution sequences of observable multiactions are totally ordered by precedence*, in contrast to the partial order semantics (*true concurrency*) exhibited by PBC. But notice that we still have a certain degree of parallelism at the low level, because we are dealing with multiactions (multisets of actions), and thus, a process evolves by executing a multiset of actions in a single step.

In our previous paper [12] our goal was to keep as far as possible, both the syntax and the operational behaviour of the operators of PBC, but extending them in the adequate way, according to the stochastic interpretation. For instance, for the new multiactions generated by the application of the synchronization operator, we needed to select a new parameter for the corresponding exponential distribution and several criteria could be applied to do it. We concluded that a good option could be to take as the value of the parameter the least of the values involved in the synchronization, which at the intuitive level could be interpreted as taking the delay of the slowest of the values involved in the synchronization as the delay of the new multiaction (capturing the synchronization). But with this definition we found some problems in order to define an adequate stochastic equivalence relation, which motivated a new proposal for the synchronization taking into account the so-called *conflict rates* (see [11]). However, this is not the only change required to obtain a congruence relation, and we show in this paper that we need to introduce some new rules in the operational semantics of sPBC, as well as a new special kind of transitions. With the new rules we will be able to distinguish between processes that reach their final state and those that are not able to do that. With the new special transitions we will be able to capture those pairs of stochastic multiactions that could be executed in parallel. They will be called *ghosts transitions*, since they will not be used to evolve in the labelled transition system in the usual way; they are only included in order to identify that information.

The paper is structured as follows: in Section 2 we present an overview of the syntax and the operational semantics of finite sPBC. In Section 3 we define the congruence relation, as well as some previous equivalence relations, in our way to obtain this congruence. Finally, Section 4 contains some conclusions and the future work.

2 Finite Stochastic Petri Box Calculus

2.1 Syntax

From now onwards we will use the following notation:

- $\mathcal{A}$ will be a countable set of action names. $\forall\, a \in \mathcal{A}$, $\exists\, \widehat{a} \in \mathcal{A}$, such that $a \neq \widehat{a}$ and $\widehat{\widehat{a}} = a$, as in CCS [14]. Letters $a, b, \widehat{a}, ...$ will be used to denote the elements of $\mathcal{A}$.
- $\mathcal{L} = \mathcal{B}(\mathcal{A})$, will represent the set of finite multisets of elements in $\mathcal{A}$ (*multiactions*).
- We define the alphabet of $\alpha \in \mathcal{L}$ by: $A(\alpha) = \{a \in \mathcal{A} \,|\, \alpha(a) > 0\}$
- Relabelling functions $f : \mathcal{A} \rightarrow \mathcal{A}$, which preserve conjugates, i.e.: $\forall a \in \mathcal{A},\ f(\widehat{a}) = \widehat{f(a)}$. We only will consider bijective functions.
- We define the set of stochastic multiactions by $\mathcal{SL} = \{<\alpha, r> \ |\, \alpha \in \mathcal{L}\, , r \in \mathbb{R}^+\}$. We allow the same multiaction $\alpha \in \mathcal{L}$ to have different stochastic rates in the same specification.
- Synchronization of multiactions:
  $\alpha \oplus_a \beta =_{def} \gamma\,,\quad \text{where: } \gamma(b) = \begin{cases} \alpha(b) + \beta(b) - 1 & \text{if } b = a \vee b = \widehat{a} \\ \alpha(b) + \beta(b) & \text{otherwise} \end{cases}$
  which is only applicable when $a \in \alpha$ and $\widehat{a} \in \beta$, or $\widehat{a} \in \alpha$ and $a \in \beta$.

Static s-expressions are used to describe the structure of a concurrent system, while dynamic s-expressions describe the current state of a system (they correspond to unmarked and marked Petri nets, respectively). As a system evolves by executing multiactions, the dynamic s-expression describing its current state changes; this is captured by means of both overbars and underbars that decorate the static s-expression. Static s-expressions of sPBC are those defined by the following BNF expression:

$$E \ ::= \ <\alpha, r> \ \mid E; E \mid E \,\Box\, E \mid E \,\|\, E \mid E[f] \mid E\, sy\, a \mid E\, rs\, a \ \mid \ [a : E]$$

where $<\alpha, r>\in \mathcal{SL}$ stands for the *basic multiaction* (simultaneous execution of all the actions in α, after a delay that follows a negative exponential distribution with parameter r). $E_1 \,;\, E_2$ stands for the sequential execution of E_1 and E_2, while $E_1 \,\Box\, E_2$ is a choice, $[f]$ is the relabelling operator, and *rs* the restriction over the single action a. The parallel operator, $\|$, represents the (disjoint) parallel execution of both components, as in PBC, i.e., there is no synchronization embedded with it. Synchronization is captured by the operator *sy*, thus the process $E\, sy\, a$ behaves in the same way as E, but it can also execute some new multiactions, generated by the synchronization of a pair of actions $(a, \hat{a})$. Finally, $[a : E]$ is a derived operator (*scoping*), which is defined by $[a : E] = (E\, sy\, a)\, rs\, a$.

We will denote static s-expressions by letters: E, F, $E_i, \ldots$, and the set of static s-expressions by *StatExpr*. However, we need to restrict the syntax of sPBC to those terms for which no parallel behaviour appears at the highest level in a choice. This restriction slightly reduces the expressiveness of the language, although we could prefix parallel operators appearing at the highest level of a choice by an empty multiaction, the rate of which could be adequately selected in order to preserve the probability of execution of the multiactions involved in the choice. Terms fulfilling this restriction will be called *regular terms*, and the operational semantics is only defined for them. This restriction is introduced in order to guarantee that the order in which the rule for the synchronization is applied does not affect the final value that we obtain for the rate of the new stochastic multiaction.

Then, a regular static s-expression E is a static s-expression of sPBC, which fulfills for all static s-expressions $E_i, i = 1, 2, 3, 4$, and $op \in \{sy,\ rs,\ [.]\}$:

$$\begin{array}{ll} E \neq (E_1 \| E_2) \Box E_3 & E \neq E_3 \Box (E_1 \| E_2) \\ E \neq op(E_1 \| E_2) \Box E_3 & E \neq E_3 \Box op(E_1 \| E_2) \\ E \neq (op(E_1 \| E_2)\,;\, E_3) \,\Box\, E_4 & E \neq E_1 \,\Box\, (op(E_2 \| E_3)\,;\, E_4) \end{array}$$

The operational semantics of sPBC is defined by means of dynamic s-expressions, which derive from the static s-expressions, but annotating them with either upper or lower bars to indicate the *active components* at each instant of time.

$$G ::= \ \overline{E} \mid \underline{E} \mid G;E \mid E;G \mid G \Box E \mid E \Box G \mid G \| G \mid G[f] \mid G\, sy\, a \mid G\, rs\, a \mid [a : G]$$

where $\overline{E}$ denotes the initial state of E, and $\underline{E}$ its final state. We will say that a dynamic s-expression is regular if the underlying static s-expression is regular. The set of regular dynamic s-expressions will be denoted by *ReDynExpr*.

2.2 Operational Semantics

The operational semantics of sPBC is defined in a very similar way to that of [3, 4, 6, 8] to define the operational semantics of PBC. We firstly present the inaction rules. They are introduced to establish the active components of a regular dynamic expression and they will capture the equivalence of regular dynamic s-expressions. Inaction rules for sPBC are those defined in Tables 1 and 2.

$\overline{E;F} \xrightarrow{\emptyset} \overline{E};F$	$\underline{E};F \xrightarrow{\emptyset} E;\overline{F}$	$E;\underline{F} \xrightarrow{\emptyset} \underline{E;F}$	
$\overline{E \Box F} \xrightarrow{\emptyset} \overline{E} \Box F$	$\overline{E \Box F} \xrightarrow{\emptyset} E \Box \overline{F}$	$\underline{E} \Box F \xrightarrow{\emptyset} \underline{E \Box F}$	$E \Box \underline{F} \xrightarrow{\emptyset} \underline{E \Box F}$
$\overline{E \| F} \xrightarrow{\emptyset} \overline{E} \| \overline{F}$	$\underline{E} \| \underline{F} \xrightarrow{\emptyset} \underline{E \| F}$	$\overline{E[f]} \xrightarrow{\emptyset} \overline{E}[f]$	$\underline{E}[f] \xrightarrow{\emptyset} \underline{E[f]}$
$\overline{E\, sy\, a} \xrightarrow{\emptyset} \overline{E}\, sy\, a$	$\underline{E}\, sy\, a \xrightarrow{\emptyset} \underline{E\, sy\, a}$	$\overline{E\, rs\, a} \xrightarrow{\emptyset} \overline{E}\, rs\, a$	$\underline{E}\, rs\, a \xrightarrow{\emptyset} \underline{E\, rs\, a}$

Table 1: Inaction rules (I)

Definition 1 Whenever G and G' are in *ReDynExpr*, such that $G \xrightarrow{\emptyset} G'$, we say that G' is an immediate derivation of G. Furthermore, when $G \xrightarrow{\emptyset} G_1 \xrightarrow{\emptyset} G_2 \xrightarrow{\emptyset} \ldots \xrightarrow{\emptyset} G'$, we will say that G' is a *derivation* of G. $\Box$

Definition 2 We say that a regular dynamic s-expression G is *operative* if it has no derivations. We will denote the set of all the operative regular dynamic s-expressions by *OpReDynExpr*. $\Box$

$\dfrac{\forall op \in \{;\,,\,\Box\},\ G \xrightarrow{\emptyset} G'}{op(G,E) \xrightarrow{\emptyset} op(G',E)}$	$\dfrac{\forall op \in \{;\,,\,\Box\},\ G \xrightarrow{\emptyset} G'}{op(E,G) \xrightarrow{\emptyset} op(E,G')}$	$\dfrac{G \xrightarrow{\emptyset} G'}{G[f] \xrightarrow{\emptyset} G'[f]}$
$\dfrac{G_1 \xrightarrow{\emptyset} G_1'}{G1 \parallel G_2 \xrightarrow{\emptyset} G_1' \parallel G_2}$	$\dfrac{G_2 \xrightarrow{\emptyset} G_2'}{G1 \parallel G_2 \xrightarrow{\emptyset} G_1 \parallel G_2'}$	$\dfrac{\forall op \in \{sy\,,\, rs\},\ G \xrightarrow{\emptyset} G'}{op(G,a) \xrightarrow{\emptyset} op(G',a)}$

Table 2: Inaction rules (II)

Definition 3 We define the structural equivalence relation for regular dynamic s-expressions as follows:

$$\equiv\ =_{def}\ (\xrightarrow{\emptyset} \cup \xleftarrow{\emptyset})^*$$

We denote the class of G with respect to $\equiv$ by $[G]_{\equiv}$. $\Box$

Rules defining the stochastic transitions are those presented in Table 3, together with those corresponding to the synchronization operator, which will be described in detail later. We assume that all dynamic s-expressions that appear on the left-hand sides of each transition in the rules are regular and operative.

(B)	$\dfrac{}{\overline{<\alpha,r>} \xrightarrow{<\alpha,r>} \underline{<\alpha,r>}}$	**(S1)**	$\dfrac{G \xrightarrow{<\alpha,r>} G'}{G;F \xrightarrow{<\alpha,r>} G';F}$	**(S2)**	$\dfrac{H \xrightarrow{<\alpha,r>} H'}{E;H \xrightarrow{<\alpha,r>} E;H'}$
(Rs)	$\dfrac{G \xrightarrow{<\alpha,r>} G'}{G\,rs\,a \xrightarrow{<\alpha,r>} G'\,rs\,a}\ a,\hat{a} \notin A(\alpha)$	**(Re)**	$\dfrac{G \xrightarrow{<\alpha,r>} G'}{G[f] \xrightarrow{<f(\alpha),r>} G'[f]}$	**(E1)**	$\dfrac{G \xrightarrow{<\alpha,r>} G'}{G\Box F \xrightarrow{<\alpha,r>} G'\Box F}$
(E2)	$\dfrac{H \xrightarrow{<\alpha,r>} H'}{E\Box H \xrightarrow{<\alpha,r>} E\Box H'}$	**(C1)**	$\dfrac{G \xrightarrow{<\alpha,r>} G'}{G\parallel H \xrightarrow{<\alpha,r>} G'\parallel H}$	**(C2)**	$\dfrac{H \xrightarrow{<\alpha,r>} H'}{G\parallel H \xrightarrow{<\alpha,r>} G\parallel H'}$

Table 3: Rules defining the stochastic transitions (I)

Observe from Table 3 that we are considering a total order semantics, and thus in order to define the semantics of the synchronization we need to capture all the possible sets of bags of stochastic multiactions that can be executed concurrently for any operative regular dynamic s-expression.

Definition 4 We define $BC\ :\ OpReDynExpr \longrightarrow \mathcal{P}(\mathcal{B}(SL))$, as follows:

- If $G \in OpReDynExpr$ is final (i.e. G=$\underline{E}$), we take $BC(G) = \emptyset$.
- If $G \in OpReDynExpr$ is not final:
  - $BC(\overline{<\alpha,r>}) = \{\{<\alpha,r>\}\}$
  - If $\gamma \in BC(G)$, then for every non-empty submultiset γ_i of γ we have $\gamma_i \in BC(G)$.
  - If $\gamma \in BC(G)$, then: $\gamma \in BC(G;E)$, $\gamma \in BC(E;G)$, $\gamma \in BC(E\Box G), \gamma \in BC(G\Box E)$, $\gamma \in BC(G\,rs\,a)$ (when $a, \hat{a} \notin A(\gamma)$), $\gamma \in BC(G\,sy\,a), f(\gamma) \in BC(G[f])$.

– If $\gamma_1 \in BC(G)$, $\gamma_2 \in BC(H)$, then $\gamma_1 + \gamma_2 \in BC(G \| H)$.

– $\gamma \in BC(G\, sy\, a)$, and $<\alpha, r_1>$, $<\beta, r_2> \in \gamma$, with $a \in A(\alpha)$, and $\widehat{a} \in A(\beta)$, then: $\gamma' \in BC(G\, sy\, a)$, where: $\gamma' = (\gamma + \{<\alpha \oplus_a \beta, R>\}) \setminus \{<\alpha, r_1>, <\beta, r_2>\}$ and R is the rate for the new stochastic multiaction, which will be defined later (see rule *Sy2* in Table 4).

$\square$

In order to define the new rates for the stochastic multiactions generated by synchronizations we need to identify conflicts. Concretely, we define for every operative regular dynamic s-expression G the multiset of associated conflicts for every instance of a stochastic multiaction $<\alpha, r>_i$ executable from G. We will denote this multiset of conflicts by $\mathit{Conflict}(G, <\alpha, r>_i)$, although we will omit the subindex i if it is clear which instance of $<\alpha, r>$ we are considering.

Definition 5 We define the following partial function:

$$\mathit{Conflict} : \mathit{OpReDynExpr} \times \mathcal{SL} \longrightarrow \mathcal{B}(\mathcal{SL})$$

$\mathit{Conflict}(G, <\alpha, r>_i)$ defines the multiset of stochastic multiactions $<\alpha, r'>$ in *conflict* with G for the instance i of the stochastic multiaction $<\alpha, r>$, which must be executable from G. We define this function in a structural way, but let us observe that in this definition G cannot be final ($G \not\equiv \underline{E}$), because we are assuming that $<\alpha, r>$ is executable from G.

1. $\mathit{Conflict}\,(\overline{<\alpha, r>}, <\alpha, r>) = \{<\alpha, r>\}$

2. If $<\alpha, r>$ is executable from G, and $C = \mathit{Conflict}\,(G, <\alpha, r>)$, then:

   (a) $\mathit{Conflict}\,(G; E, <\alpha, r>) = \mathit{Conflict}\,(E; G, <\alpha, r>) = C$,

   (b) $\mathit{Conflict}\,(G \| H, <\alpha, r>) = \mathit{Conflict}\,(H \| G, <\alpha, r>) = C$,

   (c) If $a, \hat{a} \notin A(\alpha)$, then $\mathit{Conflict}\,(G\, rs\, a, <\alpha, r>) = C$,

   (d) For a bijective function f, $\mathit{Conflict}\,(G[f], <f(\alpha), r>) = f(C)$,

   (e) For the choice operator we need to distinguish the following cases:
   - If $G \not\equiv \overline{E}$, $\mathit{Conflict}\,(G \,\square\, F, <\alpha, r>) = \mathit{Conflict}\,(F \,\square\, G, <\alpha, r>) = C$
   - If $G \equiv \overline{E}$, $\mathit{Conflict}\,(G \,\square\, F, <\alpha, r>) = \mathit{Conflict}\,(F \,\square\, G, <\alpha, r>) =$
     $C + \{\!\mid <\alpha, r_j> \mid \exists\, H_i \in \mathit{OpReDynExpr}\,,\ H_i \equiv \overline{F} \text{ and } H_i \stackrel{<\alpha, r_j>}{\longrightarrow} H_i' \mid\!\}$

   (f) $\mathit{Conflict}\,(G\, sy\, a, <\alpha, r>) = C$,

3. Let $<\alpha_1, r_1>, <\alpha_2, r_2> \in BC\,(G\, sy\, a)$, $a \in A(\alpha_1)$, $\hat{a} \in A(\alpha_2)$ and $G\, sy\, a \stackrel{<\alpha_1 \oplus_a \alpha_2, R_{12}>}{\longrightarrow} G'\, sy\, a$ obtained by applying the rule *Sy2*. Then:
   $\mathit{Conflict}\,(G\, sy\, a, <\alpha_1 \oplus_a \alpha_2\,,\, R_{12}>) =$

   $$\{\!\mid <\alpha_1 \oplus_a \alpha_2\,,\, R_{ij}> \mid <\alpha_1, r_i> \in C_1,\ <\alpha_2, r_j> \in C_2, \text{ with}$$
   $$R_{ij} = \frac{r_i}{cr(G\, sy\, a, <\alpha_1, r_1>)} \frac{r_j}{cr(G\, sy\, a, <\alpha_2, r_2>)} \cdot \min_{i=1,2} \{cr(G\, sy\, a, <\alpha_i, r_i>)\} \mid\!\}$$

   considering: $C_i = \mathit{Conflict}(G\, sy\, a, <\alpha_i, r_i>)$, $i = 1, 2$, and $cr(G, <\alpha, r>_i)$ (*conflict rate*) for G and $<\alpha, r>_i$ defined by:

$$cr(G, <\alpha, r>_i) = \sum_{<\alpha, r_j>_k \in \mathit{Conflict}\,(G, <\alpha, r>_i)} r_j$$

$\square$

Notice that $\mathit{Conflict}(G, <\alpha, r>_i)$ is a partial function, it is only defined if $<\alpha, r>_i$ is executable from G, and it is well defined. The only case requiring some explanations is that of $G\ sy\ a$ with a stochastic multiaction obtained from a synchronization: $<\alpha_1 \oplus_a \alpha_2, R_{12}>$, since we have to compute $cr(G sy\ a, <\alpha_i, r_i>)$, for $i = 1, 2$, and we need the multiset $\mathit{Conflict}(G\ sy\ a, <\alpha_i, r_i>)$ for that. Thus, this is a recursive definition, and the base case is that of a stochastic multiaction that was executable from G.

Rules for the synchronization are shown in Table 4. Observe that we take as rate of the new stochastic multiaction the minimum of the conflict rates of $<\alpha_1, r_1>$, $<\alpha_2, r_2>$, weighted by a factor, which is introduced in order to guarantee that the equivalence relation that will be introduced in Def. 11 is a congruence (see Example 5). For short, we will denote the synchronization of two stochastic multiactions $<\alpha_1, r_1>$, $<\alpha_2, r_2>$ as follows: $<\alpha_1, r_1> \oplus_a <\alpha_2, r_2>$.

(Sy1) $$\frac{G \stackrel{<\alpha,r>}{\longrightarrow} H}{G\ sy\ a \stackrel{<\alpha,r>}{\longrightarrow} H\ sy\ a}$$

(**Sy2**) Let $\{<\alpha_1, r_1>, <\alpha_2, r_2>\} \in BC(G\ sy\ a)$, $a \in A(\alpha_1)$, $\hat{a} \in A(\alpha_2)$, then

$$\frac{G\ sy\ a \stackrel{<\alpha_1,r_1>}{\longrightarrow} G_1\ sy\ a\ (\stackrel{\emptyset}{\longrightarrow})^*\ G_1^*\ sy\ a \stackrel{<\alpha_2,r_2>}{\longrightarrow} G_{12}\ sy\ a}{G\ sy\ a \stackrel{<\alpha_1 \oplus_a \alpha_2\, ,\, R>}{\longrightarrow} G_{12}\ sy\ a}$$

$$R = \frac{r_1}{cr(G\ sy\ a, <\alpha_1, r_1>)} \frac{r_2}{cr(G\ sy\ a, <\alpha_2, r_2>)} \cdot \min_{i=1,2} \{cr(G\ sy\ a, <\alpha_i, r_i>)\}$$

Table 4: Rules for the synchronization operator

Example 1 Let us consider the following operative regular dynamic s-expression:

$$G = ((\overline{<\{a, \hat{a}\}, r_1>\Box}\ <\{a, \hat{a}\}, r_2>) \| (\overline{<\{a, \hat{a}\}, r_3>\Box}\ <\{b\}, r_4>))\ sy\ a$$

From the previous definition we have:

$$\begin{aligned} \mathit{Conflict}(G, <\{a, \hat{a}\}, r_1>) &= \{<\{a, \hat{a}\}, r_1>, <\{a, \hat{a}\}, r_2>\} \\ \mathit{Conflict}(G, <\{a, \hat{a}\}, r_2>) &= \{<\{a, \hat{a}\}, r_1>, <\{a, \hat{a}\}, r_2>\} \\ \mathit{Conflict}(G, <\{a, \hat{a}\}, r_3>) &= \{<\{a, \hat{a}\}, r_3>\} \end{aligned}$$

Therefore:

$$<\{a, \hat{a}\}, r_1> \oplus_a <\{a, \hat{a}\}, r_3> = <\{a, \hat{a}\}, R_1>$$
$$<\{a, \hat{a}\}, r_2> \oplus_a <\{a, \hat{a}\}, r_3> = <\{a, \hat{a}\}, R_2>$$

where $R_1 = \frac{r_1}{r_1 + r_2} \cdot \frac{r_3}{r_3} \cdot min\{r_1 + r_2, r_3\}$, and $R_2 = \frac{r_2}{r_1 + r_2} \cdot \frac{r_3}{r_3} \cdot min\{r_1 + r_2, r_3\}$

Then, we may obtain now: $\mathit{Conflict}(G, <\{a, \hat{a}\}, R_1>) = \{<\{a, \hat{a}\}, R_1>, <\{a, \hat{a}\}, R_2>\}$, and $cr(G, <\{a, \hat{a}\}, R_1>) = min\{r_1 + r_2, r_3\}$. $\Box$

Definition 6 Let $G \in \mathit{ReDynExpr}$ be, we define the set of all dynamic s-expressions that can be derived from $[G]_{\equiv}$ by:

$$[G\rangle = \{G\} \cup \{H' \in \mathit{ReDynExpr} \mid \exists <\alpha_1, r_1>, \ldots, <\alpha_n, r_n> \in \mathcal{SL} \text{ with } G \equiv G' \stackrel{<\alpha_1,r_1>}{\longrightarrow} G_1 \equiv G_1' \stackrel{<\alpha_2,r_2>}{\longrightarrow} \ldots G_{n-1} \equiv G_{n-1}' \stackrel{<\alpha_n,r_n>}{\longrightarrow} H \equiv H'\}$$

□

Let us now see that for any regular operative dynamic s-expression G of finite PBC, and any bag of concurrent stochastic multiactions of G, then any serialization of it can be executed from G.

Lemma 1 Given a regular operative dynamic s-expression G, $\gamma \in BC(G)$, and any serialization of the stochastic multiactions of γ: $<\alpha_1, r_1> \ldots <\alpha_n, r_n>$, there exists a transition sequence:

$$G \stackrel{<\alpha_1,r_1>}{\longrightarrow} G_1 (\stackrel{\emptyset}{\longrightarrow})^* G_1^* \stackrel{<\alpha_2,r_2>}{\longrightarrow} G_2 (\stackrel{\emptyset}{\longrightarrow})^* G_2^* \stackrel{<\alpha_3,r_3>}{\longrightarrow} \ldots (\stackrel{\emptyset}{\longrightarrow})^* G_{n-1}^* \stackrel{<\alpha_n,r_n>}{\longrightarrow} G'$$

Moreover, all dynamic s-expressions G' thus obtained are equivalent with respect to $\equiv$.

Proof: By structural induction on the syntax of G, the base case (a simple stochastic multiaction) is trivial. A simple application of the induction hypothesis solves the choice, sequential composition, restriction and relabelling. For the parallel operator the only case requiring some comments is when $\gamma = \gamma_1 + \gamma_2$, with $\gamma_1 \in BC(G_1)$, $\gamma_2 \in BC(G_2)$, but in this case we may apply the induction hypothesis to both of them, to get two sequences of transitions which can be combined in the adequate way, because the evolutions for both components are independent.

The case of the synchronization operator, i.e., when $G = G_1 \, sy \, a$, is somewhat more involved, and we need to distinguish the different cases that may occur:

- If $\gamma \in BC(G_1)$, the result is immediate, by applying the induction hypothesis.
- If $\gamma = (\gamma_1 + \{<\alpha_1, r_1> \oplus_a <\alpha_2, r_2>\}) \setminus \{<\alpha_1, r_1>, <\alpha_2, r_2>\}$, with $\gamma_1 \in BC(G)$, and $a \in A(\alpha_1), \widehat{a} \in A(\alpha_2)$, then it could be the case that $\gamma_1 \in BC(G_1)$ or it could also have been obtained in the same way, by joining some multiactions; but this procedure cannot be applied infinitely, and therefore we may apply a new induction on the number of times that we have applied the synchronization of actions. So, our base case is $\gamma_1 \in BC(G_1)$, for which we may apply the hypothesis of the first induction, considering a serialization in which $<\alpha_1, r_1>$ and $<\alpha_2, r_2>$ appear one before the other. Although this sequence starts from G_1, the same could be obtained starting from $G_1 \, sy \, a$ (rule *Sy1*), and then, by applying rule *Sy2*, we get a sequence of transitions that corresponds to the given serialization of γ. In fact, it should be noted that the order of the remaining stochastic multiactions in γ_1 is unimportant, and thus, for any serialization of γ we may obtain the corresponding transition sequence.

  For the inductive case, we now have, as hypothesis, the property holding for $\gamma_1 \in BC(G \, sya)$, which allows us to argue along the same lines as in the base case, by considering a serialization of γ_1. The only difference is that in this case the sequence of rules that we obtain starts directly from $G_1 \, sy \, a$, instead of from G_1.

□

Furthermore, for any serialization of $\gamma \in BC(G)$, the multiset of conflicts for any stochastic multiaction of γ is preserved along its execution. We need the following previous lemma in order to prove this fact.

Lemma 2 Let $G \in \mathit{OpReDynExpr}$ be and $\gamma = \{<\alpha_1, r_1>, <\alpha_2, r_2>\} \in BC(G)$, with:

$$G \stackrel{<\alpha_1,r_1>}{\longrightarrow} H(\stackrel{\emptyset}{\longrightarrow})^* H^* \stackrel{<\alpha_2,r_2>}{\longrightarrow} J\ .$$

Then: $\mathit{Conflict}(G, <\alpha_2, r_2>) = \mathit{Conflict}(H^*, <\alpha_2, r_2>)$.

Proof: By structural induction on the syntax of G:

- **Base case:** $G = G_1 \parallel G_2$, where $G_1, G_2 \in \mathit{OpReDynExpr}$, $<\alpha_1, r_1>$ is executable from G_1 and $<\alpha_2, r_2>$ is executable from G_2. Thus:

$$G = G_1 \parallel G_2 \stackrel{<\alpha_1,r_1>}{\longrightarrow} G'_1 \parallel G_2(\stackrel{\emptyset}{\longrightarrow})^* G'^*_1 \parallel G_2 \stackrel{<\alpha_2,r_2>}{\longrightarrow} G'^*_1 \parallel G'_2$$

  Taking $H = G'_1 \parallel G_2$, $H^* = G'^*_1 \parallel G_2$ and $J = G'^*_1 \parallel G'_2$, we obtain:

$$\begin{array}{c}\mathit{Conflict}(G_1 \parallel G_2, <\alpha_2, r_2>) = \mathit{Conflict}(G_2, <\alpha_2, r_2>) = \\ \mathit{Conflict}(G'^*_1 \parallel G_2, <\alpha_2, r_2>) = \mathit{Conflict}(H^*, <\alpha_2, r_2>)\end{array}$$

- **General case:** A simple application of the induction hypothesis solves the sequential composition, restriction or relabelling. For the choice operator, i.e., either $G = G' \,\Box\, F$ or $G = F \,\Box\, G'$, notice that $G' \not\equiv \overline{E}$, for a certain E, due to the syntactical restriction that we have introduced, because we are considering a parallel behaviour in G. Then, we just need to apply the induction hypothesis for G'. Let us now consider $G = G'\ sy\ a$, this case is a bit more involved, and we need to distinguish the following cases:
  - If $\gamma \in BC(G')$: immediate, by applying the induction hypothesis.
  - If $\gamma \notin BC(G')$: In this case at least one of the involved stochastic multiactions has been obtained by using rule $Sy2$. Then, we can proceed by induction on the number of times that this rule has been applied. For simplicity, we only consider the case in which $<\alpha_2, r_2>$ has been obtained by applying $Sy2$, (for $<\alpha_1, r_1>$ we could reason in the same way, and if both have been obtained by using that rule, we could apply a double induction):
    - **Base case:** $<\alpha_2, r_2> = <\alpha'_2, r'_2> \oplus_a <\alpha''_2, r''_2>$

      Then, $\{<\alpha_1, r_1>, <\alpha'_2, r'_2>, <\alpha''_2, r''_2>\} \in BC(G')$, and

      $G = G'\ sy\ a \stackrel{<\alpha_1,r_1>}{\longrightarrow} H'\ sy\ a(\stackrel{\emptyset}{\longrightarrow})^* H'^*\ sy\ a = H^* \stackrel{<\alpha'_2,r'_2>\oplus_a<\alpha''_2,r''_2>}{\longrightarrow} J'\ sy\ a = J$

      We apply the induction hypothesis for $\{<\alpha_1, r_1>, <\alpha'_2, r'_2>\}$ and $\{<\alpha_1, r_1>, <\alpha''_2, r''_2>\}$ from G' to obtain:

$$\begin{array}{l}\mathit{Conflict}(G, <\alpha'_2, r'_2>) = \mathit{Conflict}(G', <\alpha'_2, r'_2>) = \mathit{Conflict}(H'^*, <\alpha'_2, r'_2>), \\ \mathit{Conflict}(G, <\alpha'_2, r''_2>) = \mathit{Conflict}(G', <\alpha''_2, r''_2>) = \mathit{Conflict}(H'^*, <\alpha''_2, r''_2>)\end{array}$$

      Therefore, by definition of *Conflict*:

$$\mathit{Conflict}(G, <\alpha'_2, r'_2> \oplus_a <\alpha''_2, r''_2>) = \mathit{Conflict}(H^*, <\alpha'_2, r'_2> \oplus_a <\alpha''_2, r''_2>)$$

* **General case:** We are now supposing that:

$$<\alpha_2, r_2> = <\alpha_2', r_2'> \oplus_a <\alpha_2'', r_2''>$$

where $<\alpha_2'', r_2''>$ is obtained by applying $Sy2$ n-1 times. Then, by the internal induction hypothesis:

$$\mathit{Conflict}(G, <\alpha_2'', r_2''>) = \mathit{Conflict}(H^*, <\alpha_2'', r_2''>)$$

and using the external induction hypothesis we may conclude:

$$\begin{array}{l} \mathit{Conflict}(G, <\alpha_2', r_2'>) = \mathit{Conflict}(G', <\alpha_2', r_2'>) = \\ \mathit{Conflict}(H'^*, <\alpha_2', r_2'>) = \mathit{Conflict}(H^*, <\alpha_2', r_2'>) \end{array}$$

Therefore:

$$\mathit{Conflict}(G, <\alpha_2', r_2'> \oplus_a <\alpha_2'', r_2''>) = \mathit{Conflict}(H^*, <\alpha_2', r_2'> \oplus_a <\alpha_2'', r_2''>)$$

□

Corollary 1 Let $G \in \mathit{OpReDynExpr}$ and $\gamma = \{<\alpha_1, r_1>, <\alpha_2, r_2>, \ldots, <\alpha_n, r_n>\} \in BC\,(G)$, with:

$$G \stackrel{<\alpha_1,r_1>}{\longrightarrow} G_1 (\stackrel{\emptyset}{\longrightarrow})^* G_1^* \stackrel{<\alpha_2,r_2>}{\longrightarrow} G_2 (\stackrel{\emptyset}{\longrightarrow})^* G_2^* \stackrel{<\alpha_3,r_3>}{\longrightarrow} \ldots (\stackrel{\emptyset}{\longrightarrow})^* G_{n-1}^* \stackrel{<\alpha_n,r_n>}{\longrightarrow} G'$$

Then: $\mathit{Conflict}\,(G, <\alpha_i, r_i>) = \mathit{Conflict}\,(G_{i-1}^*, <\alpha_i, r_i>)$, for $i = 2, \ldots, n$.

Proof: Immediate. □

Corollary 2 Given a regular operative dynamic s-expression G, $\gamma \in BC(G)$, and any serialization of the stochastic multiactions of γ: $<\alpha_1, r_1> . <\alpha_2, r_2> \ldots <\alpha_n, r_n>$ there exists a transition sequence:

$$G \stackrel{<\alpha_1,r_1>}{\longrightarrow} G_1 (\stackrel{\emptyset}{\longrightarrow})^* G_1^* \stackrel{<\alpha_2,r_2>}{\longrightarrow} G_2 (\stackrel{\emptyset}{\longrightarrow})^* G_2^* \stackrel{<\alpha_3,r_3>}{\longrightarrow} \ldots (\stackrel{\emptyset}{\longrightarrow})^* G_{n-1}^* \stackrel{<\alpha_n,r_n>}{\longrightarrow} G'$$

with $\mathit{Conflict}\,(G, <\alpha_i, r_i>) = \mathit{Conflict}\,(G_{i-1}^*, <\alpha_i, r_i>)$, for $i = 2, \ldots, n$.
Moreover, all dynamic s-expressions G' thus obtained are equivalent with respect to $\equiv$.

Proof: It is an immediate consequence of the previous corollary, and lemma 1. □

Let us now see how we can compute the rate of the stochastic multiaction that we obtain after a number of synchronizations.

Proposition 1 Let G be a regular operative dynamic s-expression, $\gamma = \{<\alpha_1, r_1>, <\alpha_2, r_2>, \ldots, <\alpha_n, r_n>\} \in BC\,(G)$, and a serialization of γ, for which we may apply $n-1$ times rule *Sy2*:

$$G \stackrel{<\alpha_1,r_1>}{\longrightarrow} G_1 (\stackrel{\emptyset}{\longrightarrow})^* G_1^* \stackrel{<\alpha_2,r_2>}{\longrightarrow} \ldots \stackrel{<\alpha_n,r_n>}{\longrightarrow} G_n$$

thus obtaining a single transition: $G \stackrel{<\beta,R>}{\longrightarrow} G_n$.

Then:

$$R = \Big(\prod_{k=1}^{n} \frac{r_k}{cr(G, <\alpha_k, r_k>)}\Big) \cdot \min_{k=1,\ldots,n} \{cr(G, <\alpha_k, r_k>)\}$$

Furthermore, $cr(G, <\beta, R>) = \min_{k=1,\ldots,n} \{cr(G, <\alpha_k, r_k>)\}$

Proof: We apply induction on n:

- **Base case:** ($n = 2$)

  We are applying *Sy2* once. Thus, we have: $\{<\alpha_1, r_1>, <\alpha_2, r_2>\} \in BC(G)$, and:

  $$G \overset{<\alpha_1,r_1>}{\longrightarrow} G_1(\overset{\emptyset}{\longrightarrow})^* G_1^* \overset{<\alpha_2,r_2>}{\longrightarrow} G_{12}$$

  Then:

  $$R = \frac{r_1}{cr(G, <\alpha_1, r_1>)} \cdot \frac{r_2}{cr(G, <\alpha_2, r_2>)} \cdot min\{cr(G, <\alpha_1, r_1>),\ cr(G, <\alpha_2, r_2>)\}$$

  Furthermore, applying the definition of *Conflict* for a synchronization we may easily conclude that $cr(G, <\beta, R>) = \min\limits_{i=1,2} \{cr(G, <\alpha_i, r_i>)\}$.

- **General case:** *Sy2* has been applied $n-1$ times ($n > 2$), and we must have:

  $$<\beta, R> = <\beta_1, R_1> \oplus_a <\beta_2, R_2>$$

  where $<\beta_1, R_1>$ has been obtained after $k-2$ synchronizations, and $<\beta_2, R_2>$ after $n-k$ synchronizations. Then, we may apply the induction hypothesis for both, obtaining:

  $$R_1 = \prod_{i=1}^{k-1} \frac{r_i}{cr(G, <\alpha_i, r_i>)} \cdot \min_{i=1,\dots,k-1} \{cr(G, <\alpha_i, r_i>)\}$$
  $$R_2 = \prod_{i=k}^{n} \frac{r_i}{cr(G, <\alpha_i, r_i>)} \cdot \min_{i=k,\dots,n} \{cr(G, <\alpha_i, r_i>)\}$$

  and:

  $$cr(G, <\beta_1, R_1>) = \min_{i=1,\dots,k-1} \{cr(G, <\alpha_i, r_i>)\}$$
  $$cr(G, <\beta_2, R_2>) = \min_{i=k,\dots,n} \{cr(G, <\alpha_i, r_i>)\}$$

  According to the definition for the synchronization, the rate for the new stochastic multiaction is computed as follows:

  $$R = \frac{R_1}{cr(G, <\beta_1, R_1>)} \cdot \frac{R_2}{cr(G, <\beta_2, R_2>)} \cdot \min_{i=1,2} \{\, cr(G, <\beta_i, R_i>) \,\}$$

  Hence:

  $$R = (\prod_{k=1}^{n} \frac{r_k}{cr(G, <\alpha_k, r_k>)}) \cdot \min_{k=1,\dots,n} \{cr(G, <\alpha_k, r_k>)\}$$

  And: $cr(G, <\beta, R>) = \min\limits_{k=1,\dots,n} \{cr(G, <\alpha_k, r_k>)\}$

□

Consequently, for all the possible transition sequences obtained by serialization of γ, if we can apply rule *Sy2* a number of times until reaching a single stochastic multiaction, we have that it does not matter the order in which rule *Sy2* has been applied, neither the transition sequence used, i.e., we will always obtain the same stochastic multiaction.

Corollary 3 Let G be a regular operative dynamic s-expression, $\gamma = \{<\alpha_1, r_1>, <\alpha_2, r_2>, \ldots, <\alpha_n, r_n>\} \in BC(G)$, and two permutations of the set $\{1, \cdots, n\}$: $\{i_1, \cdots, i_n\}$ and $\{j_1, \cdots, j_n\}$. Assuming that there are two serializations:

$$G \stackrel{<\alpha_{i_1}, r_{i_1}>}{\longrightarrow} G_1 (\stackrel{\emptyset}{\longrightarrow})^* G_1^* \stackrel{<\alpha_{i_2}, r_{i_2}>}{\longrightarrow} \ldots \stackrel{<\alpha_{i_n}, r_{i_n}>}{\longrightarrow} G_n$$
$$G \stackrel{<\alpha_{j_1}, r_{j_1}>}{\longrightarrow} G'_1 (\stackrel{\emptyset}{\longrightarrow})^* G_1'^* \stackrel{<\alpha_{j_2}, r_{j_2}>}{\longrightarrow} \ldots \stackrel{<\alpha_{j_n}, r_{j_n}>}{\longrightarrow} G'_n$$

From which we may apply $n-1$ times the rule $Sy2$ (for the same actions $a_1, \ldots, a_{n-1}$, possibly repeated, but the same number of times for both cases), thus obtaining a single transition for each case: $G \stackrel{<\beta_i, R_i>}{\longrightarrow} G_n$ and $G \stackrel{<\beta_j, R_j>}{\longrightarrow} G'_n$.

Then: $G_n \equiv G'_n$ and $<\beta_i, R_i> = <\beta_j, R_j>$.

Proof: We obtain $G_n \equiv G'_n$ from Corollary 2, and $\beta_i = \beta_j$ from the definition of the synchronization operator on multiactions. Finally, $R_i = R_j$ is obtained from the previous proposition. □

2.3 Labelled transition system

Definition 7 We define the labelled (multi)transition system of $G \in ReDynExpr$ in the Plotkin's style [15]: $ts(G) = (V, A, v_0)$, where:

- $V = \{[H]_\equiv \mid H \in [\, G \,\rangle\}$ is the set of states.
- $v_0 = [G]_\equiv$ is the initial state.
- A is the multiset of transitions, given by:

$$A = \{\!|\ ([H]_\equiv, <\alpha, r>, [J]_\equiv) \mid H \in [\, G \,\rangle \ \wedge\ H \stackrel{<\alpha,r>}{\longrightarrow} J \ |\!\}$$

In order to compute the number of instances of each transition $([H]_\equiv, <\alpha, r>, [J]_\equiv)$ in A, we take into account that we may have several different ways in order to derive such a stochastic transition (Corollary 3). Then, we will only consider one of these stochastic transitions, which can be made by enumerating the stochastic multiactions from left to right, in the syntax of the s-expression. Then, when we apply rule *Sy2*, the new stochastic multiaction can be annotated with the concatenation of the numbering of the corresponding stochastic multiactions involved in the synchronization, but if we detect that this numbering has been previously obtained, by a previous application of rule *Sy2*, although in different order, this new stochastic transition would not be considered.

For any labelled transition system $ts = (V, A, v_0)$, we will denote by $r(v, \alpha, w)$ the addition of the rates for all the edges labelled with α connecting v with w:

$$r(v, \alpha, w) = \sum \{\!|\ r_j \mid (v, <\alpha, r_j>, w) \in A \ |\!\}$$

When $(v, <\alpha, s>, w) \notin A$, $\forall s \in \mathbb{R}^+$, we take $r(v, \alpha, w) = 0$. □

Notice that the *race policy* governs the dynamic behaviour of the system when two or more stochastic multiactions are simultaneously enabled, i.e., when several stochastic multiactions are possible, the fastest one will win the race. Owing to the fact that we use exponential distributions and the race policy, the stochastic process associated with the evolution of every regular dynamic s-expression $\overline{E}$ is a Continuous Time Markov Chain (CTMC). The corresponding CTMC of $\overline{E}$ is obtained easily from $ts(\overline{E})$ (see [12]).

3 Equivalence Relations

We can define an isomorphism between two transitions systems, in the following way:

Definition 8 Given two labelled transitions systems (V^1, A^1, v_0^1) and (V^2, A^2, v_0^2), we say that they are isomorphic if there exists a bijection $h : V^1 \to V^2$, such that $h(v_0^1) = v_0^2$ and $\forall v, w \in V^1$, $\forall l \in \mathcal{SL} : (v, l, w) \in A^1$ if and only if $(h(v), l, h(w)) \in A^2$, with the same number of instances. □

However, as it occurs in PBC (see [6]), this simple notion of isomorphism is not a congruence in sPBC, as the following example shows:

Example 2 Let $E =< \{a\}, r_1 >$ be and $F =< \{a\}, r_1 >; < \{b\}, r_2 > rs\ b$. We may construct $ts(\overline{E})$ and $ts(\overline{F})$:

$$ts(\overline{E}) = \overline{E} \xrightarrow{< a, r_1 >} \underline{E} \qquad ts(\overline{F}) = \overline{F} \xrightarrow{< a, r_1 >} G_1$$

Both labelled transition systems are isomorphic, but if we consider:

$$\begin{aligned} E_1 = &\ < \{a\}, r_1 >; < \{c\}, r_3 > \\ F_1 = &\ (< \{a\}, r_1 >; < \{b\}, r_2 > rs\, b); < \{c\}, r_3 > \end{aligned}$$

we obtain that $ts(\overline{E_1})$ is not isomorphic to $ts(\overline{F_1})$:

$$ts(\overline{E_1}) = \overline{E_1} \xrightarrow{< a, r_1 >} H_1 \xrightarrow{< c, r_3 >} \underline{E_1} \qquad ts(\overline{F_1}) = \overline{F_1} \xrightarrow{< a, r_1 >} H_1'$$

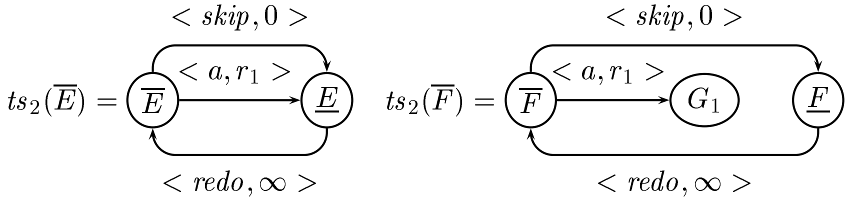
□

According to the previous example, and following the same ideas as in PBC, in order to solve this problem we have to distinguish between dynamic s-expressions that reach its final state (as $\overline{E}$), and those that will never reach its final state (as $\overline{F}$). Then, we extend the labelled transition system with two new transitions (rules of Table 5), namely *Skp* and *Rdo* (we take a rate 0 for *Skp*, which intuitively means that it will be never executable, and we take a rate ∞ for *Rdo*, which means that it is immediate).

(Skp)	$\overline{E} \stackrel{<skip,0>}{\longrightarrow} \underline{E}$
(Rdo)	$\underline{E} \stackrel{<redo,\infty>}{\longrightarrow} \overline{E}$

Table 5: Rules Skp and Rdo

We will denote by $ts_2(\overline{E})$ the new labelled transition system that we obtain considering also these two new rules. For instance, we may obtain $ts_2(\overline{E})$ and $ts_2(\overline{F})$, for E, F defined in Example 1:

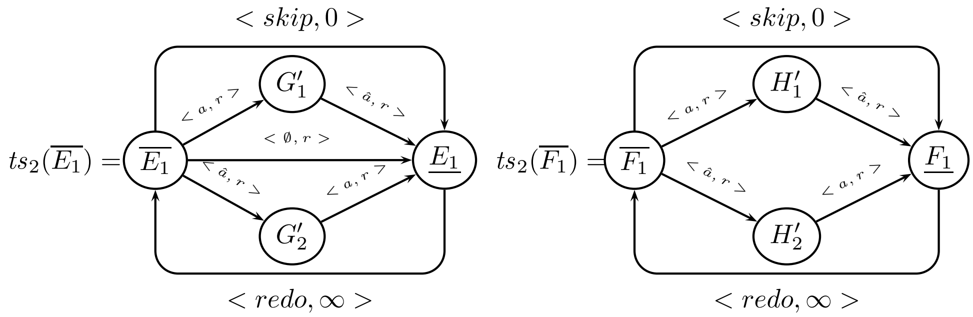


However, this definition is not yet enough to obtain a congruence in sPBC, the reason is that we have a total order semantics, and we cannot distinguish between parallelism and choices, as the following example shows:

Example 3 Let us consider:

$$E = < \{a\}, r > \, \| < \{\hat{a}\}, r >$$
$$F = < \{a\}, r >; < \{\hat{a}\}, r > \; \Box \; < \{\hat{a}\}, r >; < \{a\}, r >$$

$ts_2(\overline{E})$ and $ts_2(\overline{F})$ are isomorphic, but taking $E_1 = E \; sy \; a$ and $F_1 = F \; sy \; a$, we obtain that $ts_2(\overline{E_1})$ and $ts_2(\overline{F_1})$ are not isomorphic:



$\Box$

It seems, therefore, that we need some additional information in the labelled transition systems, which allows us to identify those transitions that have been obtained from a parallel operator, and consequently, usable for a later synchronization. Thus, we include a new kind of transitions, which we have called *ghosts*, since they cannot be used to evolve in the labelled transition system in the usual way. Our intention with these new transitions is just to annotate in the labelled transition systems the possible executions of pairs of stochastic multiactions in parallel.

Actually, we only need to capture the possible execution of two stochastic multiactions in parallel, and thus, we distinguish these transitions just annotating the stochastic multiactions over the arrow:

$$G_1 \overset{<\alpha_1, r_1> \| <\alpha_2 \, r_2>}{\dashrightarrow} G_2$$

The rule that introduces this new kind of transitions is presented in Table 6, where G is an operative regular dynamic s-expression. Notice that for each concurrent bag $\{< \alpha_1, r_1 >, < \alpha_2, r_2 >\} \in BC(G)$ there are two possible derivations:

$$G \overset{<\alpha_1, r_1> \| <\alpha_2 \, r_2>}{\dashrightarrow} G_2 \text{ and } G \overset{<\alpha_2, r_2> \| <\alpha_1 \, r_1>}{\dashrightarrow} G_2$$

Then, in the labelled transition system we will only include one of them, in the same way as we did with synchronizations.

(ghost) Let $\{<\alpha_1, r_1>, <\alpha_2, r_2>\} \in BC(G)$ $\dfrac{G \stackrel{<\alpha_1,r_1>}{\longrightarrow} G_1 \, (\stackrel{\emptyset}{\longrightarrow})^* \, G_1^* \stackrel{<\alpha_2,r_2>}{\longrightarrow} G_2}{G \stackrel{<\alpha_1,r_1>\,\|\,<\alpha_2\, r_2>}{\dashrightarrow} G_2}$

Table 6: Ghost transition rule

We now define the new transition system of $\overline{E}$, for every regular static s-expression E, taking into account the ghosts and the *Skp* and *Rdo* transitions.

Definition 9 Let E be a regular static s-expression. We define the new transition system of $\overline{E}$, $nts(\overline{E})$, by $nts(\overline{E}) = (V \cup \{[\underline{E}]_{\equiv}\}, A \cup A_{sr} \cup A_g, v_0)$, where $ts(\overline{E}) = (V, A, v_0)$, and:

- $A_{sr} = \{([\overline{E}]_{\equiv}, <skip, 0>, [\underline{E}]_{\equiv}) \, , \, ([\underline{E}]_{\equiv}, <redo, \infty>, [\overline{E}]_{\equiv})\}$
- $A_g = \{\!\mid ([H]_{\equiv}, <\alpha, r> \| <\beta, s>, [J]_{\equiv}) \mid H \in [\overline{E}\rangle \, , \, H \stackrel{<\alpha,r>\|<\beta,s>}{\dashrightarrow} J \mid\!\}$

$\square$

Using this new transition system we may introduce a new isomorphism ($\cong$):

Definition 10 For any regular static s-expressions E_1 and E_2, with:

$$nts(\overline{E_1}) = (V^{E_1} \cup \{[\underline{E_1}]_{\equiv}\}, A^{E_1} \cup A_{sr}^{E_1} \cup A_g^{E_1}, v_0^{E_1}), \text{ with } ts(\overline{E_1}) = (V^{E_1}, A^{E_1}, v_0^{E_1})$$
$$nts(\overline{E_2}) = (V^{E_2} \cup \{[\underline{E_2}]_{\equiv}\}, A^{E_2} \cup A_{sr}^{E_2} \cup A_g^{E_2}, v_0^{E_2}), \text{ with } ts(\overline{E_2}) = (V^{E_2}, A^{E_2}, v_0^{E_2})$$

We say that $E_1 \cong E_2$ if there is a bijective function $h : V^{E_1} \cup \{[\underline{E_1}]_{\equiv}\} \rightarrow V^{E_2} \cup \{[\underline{E_2}]_{\equiv}\}$, such that:

- $h(v_0^{E_1}) = v_0^{E_2}$, $h([\underline{E_1}]_{\equiv}) = [\underline{E_2}]_{\equiv}$
- $\forall v^{E_1}, w^{E_1} \in V^{E_1}$:
  - $(v^{E_1}, <\alpha, r>, w^{E_1}) \in A^{E_1}$ if and only if $(h(v^{E_1}), <\alpha, r>, h(w^{E_1})) \in A^{E_2}$, with the same number of instances.
  - $(v^{E_1}, <\alpha, r> \| <\beta, s>, w^{E_1}) \in A_g^{E_1}$ if and only if $(h(v^{E_1}), <\alpha, r> \| <\beta, s>, h(w^{E_1})) \in A_g^{E_2}$, with the same number of instances.

$\square$

Example 4 Let us consider the same processes of Example 3:

$$E = <\{a\}, r> \| <\{\hat{a}\}, r>$$
$$F = <\{a\}, r>; <\{\hat{a}\}, r> \,\square\, <\{\hat{a}\}, r>; <\{a\}, r>$$

We show $nts(\overline{E})$ and $nts(\overline{F})$ in Figure 1.

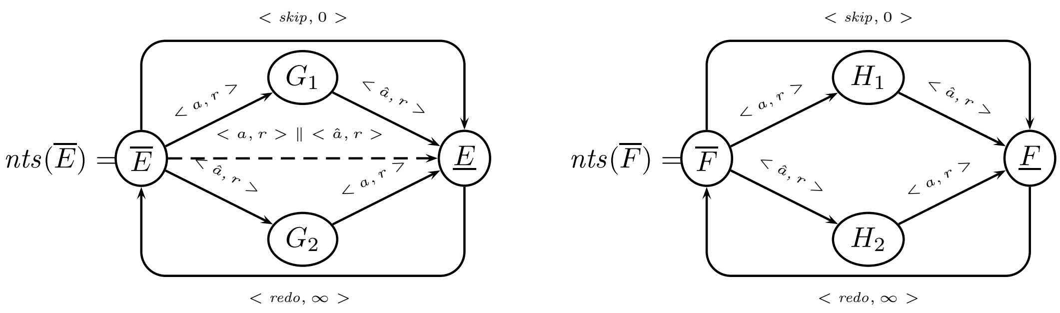


Figure 1: $nts(\overline{E})$ and $nts(\overline{F})$

Consequently, it follows that $E \not\cong F$. $\square$

This equivalence is basically the same isomorphism that we had in plain PBC. However, we would like a new equivalence, less restrictive, in which two static s-expressions are equivalent if, at the functional level they can make the same multiactions and, at the performance level they have isomorphic CTMCs. Thus, with this new equivalence we will be able to identify the following s-expressions: $<\alpha, \sum_{i=1}^{n} r_i>$ and $<\alpha, r_1> \,\square \ldots \square\, <\alpha, r_n>$.

Definition 11 Given two regular static s-expressions E_1, E_2, and:

$$
\begin{array}{l}
nts(\overline{E_1}) = (V^{E_1} \cup \{[\underline{E_1}]_\equiv\}, A^{E_1} \cup A^{E_1}_{sr} \cup A^{E_1}_{g}, v_0^{E_1}), \text{ with } ts(\overline{E_1}) = (V^{E_1}, A^{E_1}, v_0^{E_1}) \\
nts(\overline{E_2}) = (V^{E_2} \cup \{[\underline{E_2}]_\equiv\}, A^{E_2} \cup A^{E_2}_{sr} \cup A^{E_2}_{g}, v_0^{E_2}), \text{ with } ts(\overline{E_2}) = (V^{E_2}, A^{E_2}, v_0^{E_2})
\end{array}
$$

We will say that E_1 and E_2 are *stochastically equivalent* ($E_1 \sim E_2$) if and only if there is a bijective function $\phi : V^{E_1} \cup \{[\underline{E_1}]_\equiv\} \to V^{E_2} \cup \{[\underline{E_2}]_\equiv\}$, such that:

- $\phi([\overline{E_1}]_\equiv) = [\overline{E_2}]_\equiv$ and $\phi([\underline{E_1}]_\equiv) = [\underline{E_2}]_\equiv$.
- $\forall \alpha \in \mathcal{SL}$, $\forall v^1, w^1 \in V^{E_1}$, with $\phi(v^1) = v^2$, $\phi(w^1) = w^2 \in V^{E_2}$:
  - $r(v^1, \alpha, w^1) = r(v^2, \alpha, w^2)$.
  - If $(v^1, <\alpha, r_1> \parallel <\beta, r_2>, w^1) \in A_g^{E_1}$, then:
    - $(v^1, <\alpha, r_1>, x^1) \in A^{E_1}$
    - $(v^1, <\beta, r_2>, y^1) \in A^{E_1}$
    - $\exists\, s_1, s_2$ such that $(v^2, <\alpha, s_1> \parallel <\beta, s_2>, w^2) \in A_g^{E_2}$, with: $(v^2, <\alpha, s_1>, \phi(x^1)) \in A^{E_2} \wedge (v^2, <\beta, s_2>, \phi(y^1)) \in A^{E_2}$
  - If $(v^2, <\alpha, s_1> \parallel <\beta, s_2>, w^2) \in A_g^{E_2}$ then:
    - $(v^2, <\alpha, s_1>, x^2) \in A^{E_2}$
    - $(v^2, <\beta, s_2>, y^2) \in A^{E_2}$
    - $\exists\, r_1, r_2$ such that $(v^1, <\alpha, r_1> \parallel <\beta, r_2>, w^1) \in A_g^{E_1}$, with: $(v^1, <\alpha, r_1>, \phi^{-1}(x^2)) \in A^{E_1} \wedge (v^1, <\beta, r_2>, \phi^{-1}(y^2)) \in A^{E_1}$

$\square$

Example 5 The following regular static s-expressions are stochastically equivalent:

$$\begin{array}{lcl} E_1 = <\{a\},2> & \sim & E_2 = <\{a\},1> \,\square\, <\{a\},1> \\ F_1 = <\{\hat{a}\},3> & \sim & F_2 = <\{\hat{a}\},1> \,\square\, <\{\hat{a}\},2> \end{array}$$

Furthermore, we also have $E_1 \parallel F_1 \ \sim\ E_2 \parallel F_2$, and $E \sim F$, where $E = (E_1 \parallel F_1)\ sy\ a$, and $F = (E_2 \parallel F_2)\ sy\ a$. Their *new transition systems*, $nts(\overline{E})$ and $nts(\overline{F})$ are shown in Figures 2 and 3. $\square$

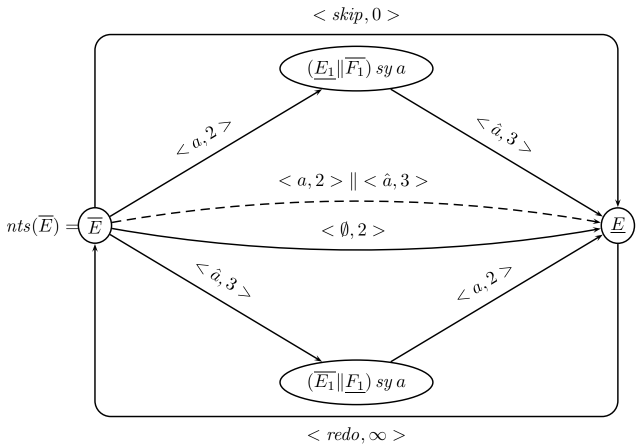


Figure 2: $nts(\overline{E})$

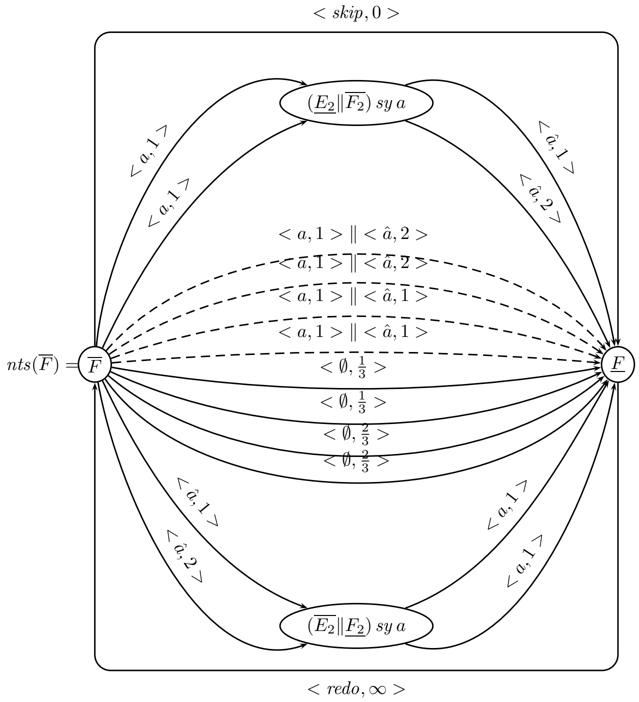


Figure 3: $nts(\overline{F})$

Ghosts transitions have been only introduced for pairs of concurrent stochastic multiactions, instead of any possible bag of them. In the following proposition we show that as a consequence of the syntactical restriction that we are considering, with this information we are able to identify all the intrinsic parallelism of two equivalent s-expressions.

Proposition 2 Let E_1, E_2, be two regular static s-expressions such that $E_1 \sim E_2$, and $H_1 \in$ *OpReDynExpr*, such that $H_1 \in [\overline{E_1}\rangle$. Then, for any $\gamma_1 = \{\!\!\{ <\alpha_i, r_i> \}\!\!\}_{i=1}^n \in BC(H_1)$ there is a regular operative dynamic s-expression $H_2 \in \phi([H_1]_\equiv)$, and a bag $\gamma_2 = \{\!\!\{ <\alpha_i, r'_i> \}\!\!\}_{i=1}^n \in BC(H_2)$.

Actually, if $([H_1]_\equiv, <\alpha_i, r_i>, [H_{1,i}]_\equiv) \in A^{E_1}$, then $([H_2]_\equiv, <\alpha_i, r'_i>, \phi([H_{1,i}]_\equiv)) \in A^{E_2}$, for $i = 1, \ldots, n$.

Proof: We apply induction on n:

- **Base case:** $(n = 2)$.

  From $\{\!\!\{ <\alpha_1, r_1>, <\alpha_2, r_2> \}\!\!\} \in BC(H_1)$, we conclude that there is a ghost transition $([H_1]_\equiv, <\alpha_1, r_1> \parallel <\alpha_2, r_2>, [H'_1]_\equiv)$ in $nts(\overline{E_1})$.

  Then, from Def. 11 we have the corresponding ghost transition in $nts(\overline{E_2})$:

  $$(\phi([H_1]_\equiv), <\alpha_1, r'_1> \parallel <\alpha_2, r'_2>, \phi([H'_1]_\equiv))$$

  with $(\phi([H_1]_\equiv), <\alpha_i, r'_i>, \phi([H_{1,i}]_\equiv)) \in A^{E_2}$, for $i = 1, 2$.

  Therefore, $\gamma_2 = \{\!\!\{ <\alpha_1, r'_1>, <\alpha_2, r'_2> \}\!\!\} \in BC(H_2)$, for some $H_2 \in \phi([H_1]_\equiv)$.

- **General case:** $(n > 2)$

  We have $\gamma_1 = \gamma_{1,1} + \{<\alpha_n, r_n>\}$, where $\gamma_{1,1} = \{\!\!\{ <\alpha_i, r_i> \}\!\!\}_{i=1}^{n-1}$. By the induction hypothesis we may find an operative dynamic s-expression $H_2 \in \phi([H_1]_\equiv)$ and $\gamma_{1,2} = \{\!\!\{ <\alpha_i, r'_i> \}\!\!\}_{1=1}^{n-1} \in BC(H_2)$, such that whether $([H_1]_\equiv, <\alpha_i, r_i>, [H_{1,i}]_\equiv) \in A^{E_1}$, then $([H_2]_\equiv, <\alpha_i, r'_i>, \phi([H_{1,i}]_\equiv)) \in A^{E_2}$, for $i = 1, \ldots, n-1$.

  Furthermore, using again Def. 11 we may conclude that for each $i \in \{1, \ldots, n-1\}$ there is an operative dynamic s-expression $H_{2,i} \in \phi([H_1]_\equiv)$, such that $\{\!\!\{ <\alpha_i, r'_i>, <\alpha_n, r'_n> \}\!\!\} \in BC(H_{2,i})$. In fact, if $([H_1]_\equiv, <\alpha_n, r_n>, [H_{1,n}]_\equiv) \in A^{E_1}$, then $([H_2]_\equiv, <\alpha_n, r'_n>, \phi([H_{1,n}]_\equiv)) \in A^{E_2}$.

  Consequently, and due to the syntactical restriction introduced, we may conclude that $\{\!\!\{ <\alpha_i, r'_i> \}\!\!\}_{i=1}^n \in BC(H_2)$, for some $H_2 \in \phi([H_1]_\equiv)$ (otherwise we would have some parallel behaviours in the arguments of a choice operator).

$\square$

In order to prove that $\sim$ is really a congruence we need three previous lemmas:

Lemma 3 Let E be a regular static s-expression, $H \in$ *OpReDynExpr*, with $H \in [\overline{E}\rangle$, and $\gamma = \{<\alpha_1, r_1>, <\alpha_2, s_1>\} \in BC(H)$.

If we have the following transitions in $nts(\overline{E})$:

$$\begin{array}{l} ([H]_\equiv, <\alpha_1, r_i>, [J]_\equiv) \in A^E,\ i = 1, \ldots, n \\ ([H]_\equiv, <\alpha_2, s_j>, [J']_\equiv) \in A^E,\ j = 1, \ldots, m \\ ([H]_\equiv, <\alpha_1, r_1> \parallel <\alpha_2, s_1>, [K]_\equiv) \in A_g^E \end{array}$$

Then, for any $i \in \{1,\ldots,n\}$, and any $j \in \{1,\ldots,m\}$, there is a ghost transition $([H]_\equiv, <\alpha_1, r_i> \parallel <\alpha_2, s_j>, [K]_\equiv) \in A_g^E$.

Proof: If $<\alpha_1, r_i>, <\alpha_2, s_j>$ cannot be executed concurrently from an operative dynamic s-expression $H_{i,j} \equiv H$, we have a conflict between them. Furthermore, $<\alpha_1, r_1>, <\alpha_1, r_i>$ are in conflict, as well as $<\alpha_2, s_1>, <\alpha_2, s_j>$, due to the syntactical restriction. Taking into account that $<\alpha_1, r_1>$ and $<\alpha_2, s_1>$ can be executed concurrently, we obtain a contradiction, because we can conclude that $<\alpha_1, r_i>$ and $<\alpha_2, s_j>$ can be executed concurrently. $\square$

Lemma 4 For any regular operative dynamic s-expression G, any stochastic multiaction $<\alpha, r>$ executable from G, and $\mathit{Conflict}(G, <\alpha, r>) = \{<\alpha, r>, <\alpha, r_1>, \ldots, <\alpha, r_n>\}$, we have:

For each $<\alpha, r_i> \in \mathit{Conflict}(G, <\alpha, r>)$ there exists $G_i \in \mathit{OpReDynExpr}$, with $G_i \equiv G$, $<\alpha, r_i>$ executable from G_i, and $\mathit{Conflict}(G_i, <\alpha, r_i>) = \mathit{Conflict}(G, <\alpha, r>)$.

Proof: By structural induction on the syntax of G.

- **Base case:** $G = \overline{<\alpha, r>}$. Immediate.
- **General case:** A simple application of the induction hypothesis solves the sequential composition, restriction, parallelism and labelling.

  For the choice operator, ($G = G_1 \square F$ or $G = F \square G_1$), we need to distinguish two cases:

  - If $G_1 \not\equiv \overline{E}$, for any regular static s-expression E, we just need to apply the induction hypothesis.
  - If $G_1 \equiv \overline{E}$ for some regular static s-expression E, we also distinguish:
    - If $<\alpha, r_i> \in \mathit{Conflict}(G_1, <\alpha, r>)$, then we can apply the induction hypothesis to conclude the property.
    - If $<\alpha, r_i> \notin \mathit{Conflict}(G_1, <\alpha, r>)$, then there exists H_i, operative, $H_i \equiv \overline{F}$, such that $<\alpha, r_i>$ is the only stochastic multiaction executable from H_i (due to the syntactical restriction), so if we consider $G_i = E \square H_i$, it follows that G_i is operative, $G_i \equiv G$ and $\mathit{Conflict}(G, <\alpha, r>) = \mathit{Conflict}(G_i, <\alpha, r_i>)$.

  The case of the synchronization operator, i.e., when $G = G_1 \, sy \, a$, is somewhat more involved, and we need to distinguish the following cases:

  - If $<\alpha, r>$ is executable from G_1, then we just need to apply the induction hypothesis.
  - Otherwise, if $<\alpha, r>$ comes from the application of rule *Sy2*, we can proceed by induction on the number of times that *Sy2* has been applied.

* **Base case:** $<\alpha, r>$ comes from the application of *Sy2* once. If we consider:

$$<\alpha, r> = <\alpha_1, r'> \oplus_a <\alpha_2, r''>$$

Then, $\{<\alpha_1, r'>, <\alpha_2, r''>\} \in BC\,(G_1\, sy\, a)$, and thus:

$$G_1\, sy\, a \overset{<\alpha_1, r'>}{\longrightarrow} G_1^1\, sy\, a (\overset{\emptyset}{\longrightarrow})^* \, G_1^{1^*}\, sy\, a \overset{<\alpha_2, r''>}{\longrightarrow} G_1^{12}\, sy\, a$$

We have:

$$C_1 \;=\; Conflict\,(G_1\, sy\, a, <\alpha_1, r'>) \;=\; Conflict\,(G_1, <\alpha_1, r'>)$$

and by corollary 1:

$$C_2 \;=\; Conflict\,(G_1\, sy\, a, <\alpha_2, r''>) \;=\; Conflict\,(G_1^{1^*}\, sy\, a, <\alpha_2, r''>) \;=\; Conflict\,(G_1^{1^*}, <\alpha_2, r''>) \;=\; Conflict\,(G_1, <\alpha_2, r''>)$$

If $<\alpha, r_i> \in Conflict\,(G, <\alpha, r>)$, then $<\alpha, r_i> = <\alpha_1, r_i'> \oplus_a <\alpha_2, r_i''>$ with $<\alpha_1, r_i'> \in C_1$ and $<\alpha_2, r_i''> \in C_2$. In G_1 we must have a parallel behaviour, in consequence, we may apply the induction hypothesis on the components of this parallel behaviour in order to conclude that there exists G_{1_i} operative, $G_{1_i} \equiv G_1$, such that $\{<\alpha_1, r_i'>, <\alpha_2, r_i''>\} \in BC\,(G_{1_i})$ with:

$$Conflict\,(G_{1_i}, <\alpha_1, r_i'>) \;=\; C_1, \qquad Conflict\,(G_{1_i}, <\alpha_2, r_i''>) \;=\; C_2$$

then, taking $G_i \;=\; G_{1_i}\, sy\, a$, it follows that G_i is operative, $G_i \equiv G$, $<\alpha, r_i>$ is executable from G_i and:

$$Conflict\,(G_i, <\alpha, r_i>) \;=\; Conflict\,(G, <\alpha, r>)$$

* **General case:** $<\alpha, r>$ has been obtained applying *Sy2* n times. Then:

$$<\alpha, r> = <\alpha_1, r'> \oplus_a <\alpha_2, r''>$$

where $<\alpha_1, r'>$ has been obtained applying *Sy2* k times, and $<\alpha_2, r''>$ has been obtained applying *Sy2* n-k-1 times. We may apply the induction hypothesis for $<\alpha_1, r'>$ and $<\alpha_2, r''>$, and conclude the proof as in the base case.

$\square$

Lemma 5 Let E_1, E_2, be two regular static s-expressions such that $E_1 \sim E_2$, $H_1, H_2' \in$ *OpReDynExpr*, such that $H_1 \in [\overline{E_1}\rangle$, $H_2 \in [\overline{E_2}\rangle$, $\phi([H_1]_\equiv) = [H_2']_\equiv$, and $<\alpha_1, r_1>$ is executable from H_1.

Then, there exists $<\alpha_1, r_2>$ executable from $H_2 \equiv H_2'$, such that:

$$cr(H_1, <\alpha_1, r_1>) = cr(H_2, <\alpha_1, r_2>)$$

Proof: We have $([H_1]_\equiv, <\alpha_1, r_1>, [J_1]_\equiv) \in A^{E_1}$, and due to the syntactical restriction for all edges $([H_1]_\equiv, <\alpha_1, r_i>, [J_1]_\equiv) \in A^{E_1}$, we have: $<\alpha_1, r_i> \in Conflict(H_1, <\alpha_1, r_1>)$. However, we may have some other edges leaving $[H_1]_\equiv$, labelled with $<\alpha_1, r^*>$, which reach nodes $[J_1']_\equiv$, with $J_1 \not\equiv J_1'$.

In order to distinguish if $<\alpha_1, r^*> \in Conflict(H_1, <\alpha_1, r_1>)$ we just need to check if there is a ghost transition: $([H_1]_\equiv, <\alpha_1, r_1> \parallel <\alpha_1, r^*>, [J_1'']_\equiv)$. If we may find such a ghost transition, $<\alpha_1, r^*>$ will not be in $Conflict(H_1, <\alpha_1, r_1>)$.

Then, we may compute $cr(H_1,<\alpha_1,r_1>)$ as follows:

$$cr(H_1,<\alpha_1,r_1>) = r([H_1]_{\equiv},\alpha_1,[J_1]_{\equiv}) + \sum_{[J'_1]_{\equiv}\in L} r([H_1]_{\equiv},\alpha_1,[J'_1]_{\equiv})$$

where $L = \{\, [J'_1]_{\equiv} \in V^{E_1} \,|\, J_1 \not\equiv J'_1,\ \exists([H_1]_{\equiv},<\alpha_1,r^*>,[J'_1]_{\equiv}) \in A^{E_1}$ and for all $([H_1]_{\equiv},<\alpha_1,r_1^*>,[J_1]_{\equiv}) \in A^{E_1}\ \not\exists\,([H_1]_{\equiv},<\alpha_1,r_1^*> \| <\alpha_1,r^*>,[J''_1]_{\equiv}) \in A_g^{E_1} \,\}$

Let us now consider $\phi([H_1]_{\equiv}) = [H'_2]_{\equiv}$ and $\phi([J_1]_{\equiv}) = [J_2]_{\equiv}$.

Since $r([H_1]_{\equiv},\alpha_1,[J_1]_{\equiv}) > 0$, we must have a stochastic multiaction $<\alpha_1,r_2>$, which is executable from H_2, with $H_2 \equiv H'_2$, and:

$$cr(H_2,<\alpha_1,r_2>) = r([H_2]_{\equiv},\alpha_1,[J_2]_{\equiv}) + \sum_{[J'_2]_{\equiv}\in L'} r([H_2]_{\equiv},\alpha_1,[J'_2]_{\equiv})$$

where $L' = \{\, [J'_2]_{\equiv} \in V^{E_2} \,|\, J_2 \not\equiv J'_2,\ \exists([H_2]_{\equiv},<\alpha_1,r^{**}>,[J'_2]_{\equiv}) \in A^{E_2}$ and for all $([H_2]_{\equiv},<\alpha_1,r_2^*>,[J_2]_{\equiv}) \in A^{E_2}\ \not\exists\,([H_2]_{\equiv},<\alpha_1,r_2^*> \| <\alpha_1,r^{**}>,[J''_2]_{\equiv}) \in A_g^{E_2} \,\}$.

Using Def. 11 we can conclude that $r([H_1]_{\equiv},\alpha_1,[J_1]_{\equiv}) = r([H_2]_{\equiv},\alpha_1,[J_2]_{\equiv})$, and $L' = \phi(L) = \{\, \phi([J'_1]_{\equiv}) \,|\, [J'_1]_{\equiv} \in L \,\}$. Therefore:

$$cr(H_1,<\alpha_1,r_1>) = cr(H_2,<\alpha_1,r_2>)$$

□

In the following theorem we prove that $\sim$ is a congruence relation:

Theorem 1 Let E_1, E_2, E be regular static s-expressions, such that $E_1 \sim E_2$, and $nts(\overline{E}) = (V^E \cup \{[\underline{E}]_{\equiv}\}, A^E \cup A_{sr}^E \cup A_g^E, v_0^E)$, $nts(\overline{E_1}) = (V^{E_1} \cup \{[\underline{E_1}]_{\equiv}\}, A^{E_1} \cup A_{sr}^{E_1} \cup A_g^{E_1}, v_0^{E_1})$, and $nts(\overline{E_2}) = (V^{E_2} \cup \{[\underline{E_2}]_{\equiv}\}, A^{E_2} \cup A_{sr}^{E_2} \cup A_g^{E_2}, v_0^{E_2})$.

Then:

- **(i)** $E_1\,;E \sim E_2\,;E$ and $E\,;E_1 \sim E\,;E_2$
- **(ii)** $E_1 \| E \sim E_2 \| E$ and $E \| E_1 \sim E \| E_2$
- **(iii)** $E_1 \,\Box\, E \sim E_2 \,\Box\, E$ and $E \,\Box\, E_1 \sim E \,\Box\, E_2$
- **(iv)** $E_1[f] \sim E_2[f]$, for a bijective relabelling function f
- **(v)** $E_1\ sy\ a \sim E_2\ sy\ a$, for $a \in \mathcal{A}$
- **(vi)** $E_1\ rs\ a \sim E_2\ rs\ a$, for $a \in \mathcal{A}$

Proof: Let ϕ be the bijective function that $\sim$ provides for us for E_1 and E_2.
(i) Let $F_1 = E_1;E$ be and $F_2 = E_2;E$. We can obtain $nts(\overline{F_1})$ and $nts(\overline{F_2})$, as follows:

$$\begin{aligned}
&nts(\overline{F_1}) = (V^{F_1} \cup \{[\underline{F_1}]_{\equiv}\}, A^{F_1} \cup A_{sr}^{F_1} \cup A_g^{F_1}, v_0^{F_1}), \quad \text{where} \quad ts(\overline{F_1}) = (V^{F_1}, A^{F_1}, v_0^{F_1})\\
&V^{F_1} = \{[H_1;E]_{\equiv} \,|\, H_1 \in [\overline{E_1}\,\rangle\,\} \cup \{[E_1;G]_{\equiv} \,|\, G \in [\overline{E}\,\rangle\,\}\\
&A^{F_1} = \{\!\!\{\ ([H_1;E]_{\equiv}, <\alpha,r>, [J_1;E]_{\equiv}) \,|\, ([H_1]_{\equiv}, <\alpha,r>, [J_1]_{\equiv}) \in A^{E_1}\ \}\!\!\} +\\
&\qquad \{\!\!\{\ ([E_1;G]_{\equiv}, <\alpha,r>, [E_1;J]_{\equiv}) \,|\, ([G]_{\equiv}, <\alpha,r>, [J]_{\equiv}) \in A^{E}\ \}\!\!\}\\
&A_{sr}^{F_1} = \{([\overline{F_1}]_{\equiv}, <skip,0>, [\underline{F_1}]_{\equiv}), ([\underline{F_1}]_{\equiv}, <redo,\infty>, [\overline{F_1}]_{\equiv})\}\\
&A_g^{F_1} = \{\!\!\{\ ([H_1;E]_{\equiv}, <\alpha,r> \| <\beta,s>, [J_1;E]_{\equiv}) \,|\, ([H_1]_{\equiv}, <\alpha,r> \| <\beta,s>, [J_1]_{\equiv}) \in A_g^{E_1}\ \}\!\!\} +\\
&\qquad \{\!\!\{\ ([E_1;G]_{\equiv}, <\alpha,r> \| <\beta,s>, [E_1;J]_{\equiv}) \,|\, ([G]_{\equiv}, <\alpha,r> \| <\beta,s>, [J]_{\equiv}) \in A_g^{E}\ \}\!\!\}\\
&v_0^{F_1} = [\overline{F_1}]_{\equiv}
\end{aligned}$$

Note that in V^{F_1}:

- $[\underline{E_1};E]_\equiv = [E_1;\overline{E}]_\equiv$ because $\underline{E_1};E \equiv E_1;\overline{E}$
- $[\overline{F_1}]_\equiv = [\overline{E_1};E]_\equiv$ because $\overline{F_1} \equiv \overline{E_1};E$
- $[\underline{F_1}]_\equiv = [E_1;\underline{E}]_\equiv$ because $\underline{F_1} \equiv E_1;\underline{E}$

Analogously:

$$
\begin{aligned}
& nts(\overline{F_2}) = (V^{F_2} \cup \{[\underline{F_2}]_\equiv\}, A^{F_2} \cup A^{F_2}_{sr} \cup A^{F_2}_{g}, v_0^{F_2}), \quad \text{where} \quad ts(\overline{F_2}) = (V^{F_2}, A^{F_2}, v_0^{F_2}) \\
& V^{F_2} = \{[H_2;E]_\equiv \mid H_2 \in [\overline{E_2}\rangle\} \cup \{[E_2;G]_\equiv \mid G \in [\overline{E}\rangle\} \\
& A^{F_2} = \{\!|\ ([H_2;E]_\equiv, <\alpha,r>, [J_2;E]_\equiv) \mid ([H_2]_\equiv, <\alpha,r>, [J_2]_\equiv) \in A^{E_2}\ |\!\} + \\
& \qquad \{\!|\ ([E_2;G]_\equiv, <\alpha,r>, [E_2;J]_\equiv) \mid ([G]_\equiv, <\alpha,r>, [J]_\equiv) \in A^{E}\ |\!\} \\
& A^{F_2}_{sr} = \{([\overline{F_2}]_\equiv, <skip,0>, [\underline{F_2}]_\equiv), ([\underline{F_2}]_\equiv, <redo,\infty>, [\overline{F_2}]_\equiv)\} \\
& A^{F_2}_{g} = \{\!|\ ([H_2;E]_\equiv, <\alpha,r> \parallel <\beta,s>, [J_2;E]_\equiv) \mid ([H_2]_\equiv, <\alpha,r> \parallel <\beta,s>, [J_2]_\equiv) \in A^{E_2}_g\ |\!\} + \\
& \qquad \{\!|\ ([E_2;G]_\equiv, <\alpha,r> \parallel <\beta,s>, [E_2;J]_\equiv) \mid ([G]_\equiv, <\alpha,r> \parallel <\beta,s>, [J]_\equiv) \in A^{E}_g\ |\!\} \\
& v_0^{F_2} = [\overline{F_2}]_\equiv
\end{aligned}
$$

With $[\underline{E_2};E]_\equiv = [E_2;\overline{E}]_\equiv$, $[\overline{F_2}]_\equiv = [\overline{E_2};E]_\equiv$, and $[\underline{F_2}]_\equiv = [E_2;\underline{E}]_\equiv$.

We define $\varphi : V^{F_1} \cup \{[\underline{F_1}]_\equiv\} \to V^{F_2} \cup \{[\underline{F_2}]_\equiv\}$, in the following way:

- If $[H_1]_\equiv \in V^{E_1}$ and $\phi([H_1]_\equiv) = [H_2]_\equiv$, then $\varphi([H_1;E]_\equiv) = [H_2;E]_\equiv$
- $\varphi([E_1;G]_\equiv) = [E_2;G]_\equiv$, where $G \in [\overline{E}\rangle$
- $\varphi([\underline{F_1}]_\equiv) = [\underline{F_2}]_\equiv$

It is immediate to check that φ is well defined, it is a bijection, and $\varphi([\overline{F_1}]_\equiv) = [\overline{F_2}]_\equiv$.

- Let us consider an edge e of A^{F_1}. We may have two cases:
  - $e = ([H_1;E]_\equiv, <\alpha,r>, [J_1;E]_\equiv)$, with $([H_1]_\equiv, <\alpha,r>, [J_1]_\equiv) \in A^{E_1}$.
    Taking $\phi([H_1]_\equiv) = [H_2]_\equiv$, $\phi([J_1]_\equiv) = [J_2]_\equiv$, then $\varphi([H_1;E]_\equiv) = [H_2;E]_\equiv$ and $\varphi([J_1;E]_\equiv) = [J_2;E]_\equiv$.
    From $E_1 \sim E_2$, we have:
    $r([H_1;E]_\equiv, \alpha, [J_1;E]_\equiv) = r([H_1]_\equiv, \alpha, [J_1]_\equiv) = r([H_2]_\equiv, \alpha, [J_2]_\equiv) = r([H_2;E]_\equiv, \alpha, [J_2;E]_\equiv)$.
    It is also immediate to check that $r([H_1;E]_\equiv, \alpha, [J_1;E]_\equiv) = 0$ if and only if $r([H_2;E]_\equiv, \alpha, [J_2;E]_\equiv) = 0$.
  - $e = ([E_1;G]_\equiv, <\alpha,r>, [E_1;J]_\equiv)$, with $([G]_\equiv, <\alpha,r>, [J]_\equiv) \in A^{E}$.
    In this case we have: $\varphi([E_1;G]_\equiv) = [E_2;G]_\equiv$, $\varphi([E_1;J]_\equiv) = [E_2;J]_\equiv$, $r([E_1;G]_\equiv, \alpha, [E_1;J]_\equiv) = r([G]_\equiv, \alpha, [J]_\equiv) = r([E_2;G]_\equiv, \alpha, [E_2;J]_\equiv)$.
    It can be easily checked that $r([E_1;G]_\equiv, \alpha, [E_1;J]_\equiv) = 0$ if and only if $r([E_2;G]_\equiv, \alpha, [E_2;J]_\equiv) = 0$.

- Let us consider an edge e of $A_g^{F_1}$. We have also two cases:
  - $e = ([H_1;E]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J_1;E]_\equiv\,)$, with $([H_1]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J_1]_\equiv\,) \in A_g^{E_1}$. Taking $\phi([H_1]_\equiv) = [H_2]_\equiv$, $\phi([J_1]_\equiv) = [J_2]_\equiv$, then $\varphi([H_1;E]_\equiv) = [H_2;E]_\equiv$ and $\varphi([J_1;E]_\equiv) = [J_2;E]_\equiv$.

    Furthermore, if $([H_1;E]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J_1;E]_\equiv\,) \in A_g^{F_1}$, with $([H_1]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J_1]_\equiv\,) \in A_g^{E_1}$, then from $E_1 \sim E_2$:

    1. $([H_1]_\equiv, <\alpha, r>, [H_1'] \equiv) \in A^{E_1}$
    2. $([H_1]_\equiv, <\beta, s>, [H_1''] \equiv) \in A^{E_1}$
    3. $\exists r', s'$ such that $([H_2]_\equiv\,, <\alpha, r'> \,\|\, <\beta, s'>, [J_2]_\equiv\,) \in A_g^{E_2}$, with: $([H_2]_\equiv\,, <\alpha, r'>, \phi([H_1']_\equiv)) \in A^{E_2}$, and $([H_2]_\equiv\,, <\beta, s'>, \phi([H_1'']_\equiv)) \in A^{E_2}$.

    Then, we conclude:

    1. $([H_1;E]_\equiv, <\alpha, r>, [H_1';E] \equiv) \in A^{F_1}$
    2. $([H_1;E]_\equiv, <\beta, s>, [H_1'';E] \equiv) \in A^{F_1}$
    3. $\exists r', s'$ such that $([H_2;E]_\equiv\,, <\alpha, r'> \,\|\, <\beta, s'>, [J_2;E]_\equiv\,) \in A_g^{F_2}$, with: $([H_2;E]_\equiv\,, <\alpha, r'>, \phi([H_1';E]_\equiv)) \in A^{F_2}$, and $([H_2;E]_\equiv\,, <\beta, s'>, \phi([H_1'';E]_\equiv)) \in A^{F_2}$.
  - $e = ([E_1;G]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [E_1;J]_\equiv\,)$ with $([G]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J]_\equiv\,) \in A_g^{E}$. This case is symmetric to the previous one.

Notice that the proof for the edges of A^{F_2} and $A_g^{F_2}$ is identical[1].

The proof of $E\,;E_1 \sim E\,;E_2$ is analogous.

(ii) Let $F_1 = E_1 \,\|\, E$ and $F_2 = E_2 \,\|\, E$ be. In this case we have:

$$nts(\overline{F_1}) = (V^{F_1} \cup \{[\underline{F_1}]_\equiv\}, A^{F_1} \cup A_{sr}^{F_1} \cup A_g^{F_1}, v_0^{F_1}), \qquad \text{where} \quad ts(\overline{F_1}) = (V^{F_1}, A^{F_1}, v_0^{F_1})$$

$$\begin{aligned}
V^{F_1} &= \{[H_1 \| G]_\equiv \quad | \quad H_1 \in [\overline{E_1}\rangle \quad \text{and} \quad G \in [\overline{E}\rangle\,\} \\
A^{F_1} &= \{\!|\ ([H_1\|G]_\equiv\,, <\alpha, r>, [J_1\|G]_\equiv\,) \,|\, ([H_1]_\equiv\,, <\alpha, r>, [J_1]_\equiv\,) \in A^{E_1} \ |\!\} + \\
&\quad\ \{\!|\ ([H_1\|G]_\equiv\,, <\alpha, r>, [H_1\|J]_\equiv\,) \,|\, ([G]_\equiv\,, <\alpha, r>, [J]_\equiv\,) \in A^{E} \ |\!\} \\
A_{sr}^{F_1} &= \{([\overline{F_1}]_\equiv, <skip, 0>, [\underline{F_1}]_\equiv), ([\underline{F_1}]_\equiv, <redo, \infty>, [\overline{F_1}]_\equiv)\} \\
A_g^{F_1} &= \{\!|\ ([H_1\|G]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J_1\|G]_\equiv\,) \,|\, ([H_1]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J_1]_\equiv\,) \in A_g^{E_1} \ |\!\} + \\
&\quad\ \{\!|\ ([H_1\|G]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [H_1\|J]_\equiv\,) \,|\, ([G]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J]_\equiv\,) \in A_g^{E} \ |\!\} + \\
&\quad\ \{\!|\ ([H_1\|G]_\equiv\,, <\alpha, r> \,\|\, <\beta, s>, [J_1\|J]_\equiv\,) \,|\, ([H_1]_\equiv\,, <\alpha, r>\ [J_1]_\equiv\,) \in A^{E_1} \text{ and} \\
&\quad\ ([G]_\equiv\,, <\beta, s>\ [J]_\equiv\,) \in A^{E} \ |\!\} \\
v_0^{F_1} &= [\overline{F_1}]_\equiv
\end{aligned}$$

With $[\overline{F_1}]_\equiv = [\overline{E_1} \| \overline{E}]_\equiv$, and $[\underline{F_1}]_\equiv = [\underline{E_1} \| \underline{E}]_\equiv$.

[1] The same occurs in the remaining operators, so in the following cases we will just describe the proofs for edges in A^{F_1} and $A_g^{F_1}$.

$$
\begin{aligned}
&nts(\overline{F_2}) = (V^{F_2} \cup \{[\underline{F_2}]_\equiv\}, A^{F_2} \cup A^{F_2}_{sr} \cup A^{F_2}_g, v_0^{F_2}), \qquad \text{where} \quad ts(\overline{F_2}) = (V^{F_2}, A^{F_2}, v_0^{F_2})\\
&V^{F_2} = \{[H_2\|G]_\equiv \quad | \quad H_2 \in [\overline{E_2}\rangle \quad \text{and} \quad G \in [\overline{E}\rangle\,\}\\
&A^{F_2} = \{\!|\ ([H_2\|G]_\equiv\,, <\alpha, r>, [J_2\|G]_\equiv\,) \,|\, ([H_2]_\equiv\,, <\alpha, r>, [J_2]_\equiv\,) \in A^{E_2}\ |\!\} +\\
&\qquad\ \ \{\!|\ ([H_2\|G]_\equiv\,, <\alpha, r>, [H_2\|J]_\equiv\,) \,|\, ([G]_\equiv\,, <\alpha, r>, [J]_\equiv\,) \in A^{E}\ |\!\}\\
&A^{F_2}_{sr} = \{([\overline{F_2}]_\equiv, <skip, 0>, [\underline{F_2}]_\equiv), ([\underline{F_2}]_\equiv, <redo, \infty>, [\overline{F_2}]_\equiv)\}\\
&A^{F_2}_g = \{\!|\ ([H_2\|G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J_2\|G]_\equiv\,) \,|\, ([H_2]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J_2]_\equiv\,) \in A^{E_2}_g\ |\!\} +\\
&\qquad\ \ \{\!|\ ([H_2\|G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [H_2\|J]_\equiv\,) \,|\, ([G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J]_\equiv\,) \in A^{E}_g\ |\!\} +\\
&\qquad\ \ \{\!|\ ([H_2\|G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J_2\|J]_\equiv\,) \,|\, ([H_2]_\equiv\,, <\alpha,r>\ [J_2]_\equiv\,) \in A^{E_2} \text{ and}\\
&\qquad\ \ ([G]_\equiv\,, <\beta,s>\ [J]_\equiv\,) \in A^{E}\ |\!\}\\
&v_0^{F_2} = [\overline{F_2}]_\equiv
\end{aligned}
$$

With $[\overline{F_2}]_\equiv = [\overline{E_2}\|\overline{E}]_\equiv$, and $[\underline{F_2}]_\equiv = [\underline{E_2}\|\underline{E}]_\equiv$.

We define $\varphi : V^{F_1} \cup \{[\underline{F_1}]_\equiv\} \to V^{F_2} \cup \{[\underline{F_2}]_\equiv\}$, in the following way:

- $\forall [H_1]_\equiv \in V^{E_1}$, $\varphi([H_1\|G]_\equiv) = [H_2\|G]_\equiv$, where $\phi([H_1]_\equiv) = [H_2]_\equiv$.
- $\varphi([\underline{F_1}]_\equiv) = [\underline{F_2}]_\equiv$.

φ is well defined, it is a bijection, and $\varphi([\overline{F_1}]_\equiv) = [\overline{F_2}]_\equiv$.

- We have again two possible cases for the edges of A^{F_1}, in both of them we can repeat a reasoning similar to that one followed in case (i), obtaining:

  - $r([H_1\|G]_\equiv\,, \alpha, [J_1\|G]_\equiv\,) = r([H_2\|G]_\equiv\,, \alpha, [J_2\|G]_\equiv\,)$.
  - $r([H_1\|G]_\equiv\,, \alpha, [H_1\|J]_\equiv\,) = r([H_2\|G]_\equiv\,, \alpha, [H_2\|J]_\equiv\,)$.

  with $\phi([H_1]_\equiv) = [H_2]_\equiv$, and $\phi([J_1]_\equiv) = [J_2]_\equiv$.

- For the edges in $A^{F_1}_g$ we have three cases:

  - $([H_1\|G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J_1\|G]_\equiv\,)$, with $([H_1]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J_1]_\equiv\,) \in A^{E_1}_g$.
  - $([H_1\|G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [H_1\|J]_\equiv\,)$, with $([G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J]_\equiv\,) \in A^{E}_g$.
  - $([H_1\|G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J_1\|J]_\equiv\,)$, with $([H_1]_\equiv\,, <\alpha,r>, [J_1]_\equiv\,) \in A^{E_1}$ and $([G]_\equiv\,, <\beta,s>, [J]_\equiv\,) \in A^{E}$.

  The two first cases are analogous to those of case (i). For the third case, from $E_1 \sim E_2$ we may find r' such that $([H_2]_\equiv\,, <\alpha,r'>, [J_2]_\equiv\,) \in A^{E_2}$, and thus: $([H_2\|G]_\equiv\,, <\alpha,r'> \,\|\, <\beta,s>, [J_2\|J]_\equiv\,) \in A^{F_2}_g$.

(iii) Let $F_1 = E_1 \,\square\, E$ and $F_2 = E_2 \,\square\, E$ be. We have:

$$
\begin{aligned}
&nts(\overline{F_1}) = (V^{F_1} \cup \{[\underline{F_1}]_\equiv\}, A^{F_1} \cup A^{F_1}_{sr} \cup A^{F_1}_g, v_0^{F_1}), \qquad \text{where} \quad ts(\overline{F_1}) = (V^{F_1}, A^{F_1}, v_0^{F_1})\\
&V^{F_1} = \{[H_1 \,\square\, E]_\equiv \quad | \quad H_1 \in [\overline{E_1}\rangle\,\} \cup \{[E_1 \,\square\, G]_\equiv \quad | \quad G \in [\overline{E}\rangle\,\}\\
&A^{F_1} = \{\!|\ ([H_1 \,\square\, E]_\equiv\,, <\alpha,r>, [J_1 \,\square\, E]_\equiv\,) \,|\, ([H_1]_\equiv\,, <\alpha,r>, [J_1]_\equiv\,) \in A^{E_1}\ |\!\} +\\
&\qquad\ \ \{\!|\ ([E_1 \,\square\, G]_\equiv\,, <\alpha,r>, [E_1 \,\square\, |J]_\equiv\,) \,|\, ([G]_\equiv\,, <\alpha,r>, [J]_\equiv\,) \in A^{E}\ |\!\}\\
&A^{F_1}_{sr} = \{([\overline{F_1}]_\equiv, <skip, 0>, [\underline{F_1}]_\equiv), ([\underline{F_1}]_\equiv, <redo, \infty>, [\overline{F_1}]_\equiv)\}\\
&A^{F_1}_g = \{\!|\ ([H_1 \,\square\, E]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J_1 \,\square\, E]_\equiv\,) \,|\, ([H_1]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J_1]_\equiv\,) \in A^{E_1}_g\ |\!\} +\\
&\qquad\ \ \{\!|\ ([E_1 \,\square\, G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [E_1 \,\square\, J]_\equiv\,) \,|\, ([G]_\equiv\,, <\alpha,r> \,\|\, <\beta,s>, [J]_\equiv\,) \in A^{E}_g\ |\!\}\\
&v_0^{F_1} = [\overline{F_1}]_\equiv
\end{aligned}
$$

Note that in V^{F_1}:

- $[\overline{F_1}]_\equiv = [\overline{E_1} \square E]_\equiv = [E_1 \square \overline{E}]_\equiv$
- $[\underline{F_1}]_\equiv = [\underline{E_1} \square E]_\equiv = [E_1 \square \underline{E}]_\equiv$

$$
\begin{aligned}
&nts(\overline{F_2}) = (V^{F_2} \cup \{[\underline{F_2}]_\equiv\}, A^{F_2} \cup A_{sr}^{F_2} \cup A_g^{F_2}, v_0^{F_2}), \quad \text{where} \quad ts(\overline{F}_2) = (V^{F_2}, A^{F_2}, v_0^{F_2}) \\
&V^{F_2} = \{[H_2 \square E]_\equiv \;\; | \;\; H_2 \in [\overline{E_2}\rangle\} \cup \{[E_2 \square G]_\equiv \;\; | \;\; G \in [\overline{E}\rangle\} \\
&A^{F_2} = \{\!\!\{ \; ([H_2 \square E]_\equiv, <\alpha, r>, [J_2 \square E]_\equiv) \,|\, ([H_2]_\equiv, <\alpha, r>, [J_2]_\equiv) \in A^{E_2} \; \}\!\!\} + \\
&\qquad \{\!\!\{ \; ([E_2 \square G]_\equiv, <\alpha, r>, [E_2 \square | J]_\equiv) \,|\, ([G]_\equiv, <\alpha, r>, [J]_\equiv) \in A^{E} \; \}\!\!\} \\
&A_{sr}^{F_2} = \{([\overline{F_2}]_\equiv, <skip, 0>, [\underline{F_2}]_\equiv), ([\underline{F_2}]_\equiv, <redo, \infty>, [\overline{F_2}]_\equiv)\} \\
&A_g^{F_2} = \{\!\!\{ \; ([H_2 \square E]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_2 \square E]_\equiv) \,|\, ([H_2]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_2]_\equiv) \in A_g^{E_2} \; \}\!\!\} + \\
&\qquad \{\!\!\{ \; ([E_2 \square G]_\equiv, <\alpha, r> \parallel <\beta, s>, [E_2 \square J]_\equiv) \,|\, ([G]_\equiv, <\alpha, r> \parallel <\beta, s>, [J]_\equiv) \in A_g^{E} \; \}\!\!\} \\
&v_0^{F_2} = [\overline{F_2}]_\equiv
\end{aligned}
$$

With $[\overline{F_2}]_\equiv = [\overline{E_2} \square E]_\equiv = [E_2 \square \overline{E}]_\equiv$, and $[\underline{F_2}]_\equiv = [\underline{E_2} \square E]_\equiv = [E_2 \square \underline{E}]_\equiv$.

We define $\varphi : V^{F_1} \cup \{[\underline{F_1}]_\equiv\} \rightarrow V^{F_2} \cup \{[\underline{F_2}]_\equiv\}$, in the following way:

- $\forall [H_1]_\equiv \in V^{E_1}$, $\varphi([H_1 \square E]_\equiv) = [H_2 \square E]_\equiv$, where $\phi([H_1]_\equiv) = [H_2]_\equiv$.
- $\varphi([E_1 \square G]_\equiv) = [E_2 \square G]_\equiv$.
- $\varphi([\underline{F_1}]_\equiv) = [\underline{F_2}]_\equiv$.

φ is well defined, it is a bijection and $\varphi([\overline{F_1}]_\equiv) = [\overline{F_2}]_\equiv$.

- Let us consider an edge e in A^{F_1}. We may have the following four cases:

  - $e = ([H_1 \square E]_\equiv, <\alpha, r>, [J_1 \square E]_\equiv)$, with $H_1 \not\equiv \overline{E_1} \vee (H_1 \equiv \overline{E_1} \wedge J_1 \not\equiv \underline{E_1})$.
  - $e = ([E_1 \square G]_\equiv, <\alpha, r>, [E_1 \square J]_\equiv)$, with $G \not\equiv \overline{E} \vee (G \equiv \overline{E} \wedge J \not\equiv \underline{E})$.
  - $e = ([H_1 \square E]_\equiv, <\alpha, r>, [J_1 \square E]_\equiv)$, with $H_1 \equiv \overline{E_1}$, and $J_1 \equiv \underline{E_1}$.
  - $e = ([E_1 \square G]_\equiv, <\alpha, r>, [E_1 \square J]_\equiv)$, with $G \equiv \overline{E}$, and $J \equiv \underline{E}$.

  The two first cases are solved using a similar reasoning to that one followed in (i) and (ii). The third and fourth cases are similar, so we only show the proof for the third one:
  $r([H_1 \square E]_\equiv, \alpha, [J_1 \square E]_\equiv) = r([\overline{E_1}]_\equiv, \alpha, [\underline{E_1}]_\equiv) + r([\overline{E}]_\equiv, \alpha, [\underline{E}]_\equiv) =$
  $r([\overline{E_2}]_\equiv, \alpha, [\underline{E_2}]_\equiv) + r([\overline{E}]_\equiv, \alpha, [\underline{E}]_\equiv) = r([\overline{E_2} \square E]_\equiv, \alpha, [\underline{E_2} \square E]_\equiv) =$
  $r([H_2 \square E]_\equiv, \alpha, [J_2 \square E]_\equiv), \forall H_2 \equiv \overline{E_2}, \forall J_2 \equiv \underline{E_2}$.

- For the edges in $A_g^{F_1}$ we have again two possible cases:

  - $([H_1 \square E]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1 \square G]_\equiv)$, with $([H_1]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1]_\equiv) \in A_g^{E_1}$.
  - $([E_1 \square G]_\equiv, <\alpha, r> \parallel <\beta, s>, [E_1 \square J]_\equiv)$, with $([G]_\equiv, <\alpha, r> \parallel <\beta, s>, [J]_\equiv) \in A_g^{E}$

Both are immediate, and they can be solved with a similar reasoning to that one followed in (i) and (ii).

(iv) Let $F_1 = E_1\,[f]$ and $F_2 = E_2\,[f]$ be. In this case $nts(\overline{F_1})$ and $nts(\overline{F_2})$ are obtained as follows:

$$\begin{array}{l}
nts(\overline{F_1}) = (V^{F_1} \cup \{[\underline{F_1}]_\equiv\}, A^{F_1} \cup A_{sr}^{F_1} \cup A_g^{F_1}, v_0^{F_1}), \quad \text{where} \quad ts(\overline{F_1}) = (V^{F_1}, A^{F_1}, v_0^{F_1}) \\
V^{F_1} = \{[H_1\,[f]]_\equiv \quad | \quad H_1 \in [\overline{E_1}\,\rangle\,\} \\
A^{F_1} = \{\!| \; ([H_1\,[f]\,]_\equiv\,, <f(\alpha), r>, [J_1\,[f]\,]_\equiv\,) \,|\, ([H_1]_\equiv\,, <\alpha, r>, [J_1]_\equiv\,) \in A^{E_1} \; |\!\} \\
A_{sr}^{F_1} = \{([\overline{F_1}]_\equiv, <skip, 0>, [\underline{F_1}]_\equiv), ([\underline{F_1}]_\equiv, <redo, \infty>, [\overline{F_1}]_\equiv)\} \\
A_g^{F_1} = \{\!| \; ([H_1\,[f]\,]_\equiv\,, <f(\alpha), r> \parallel <f(\beta), s>, [J_1\,[f]\,]_\equiv\,) \,|\, ([H_1]_\equiv\,, <\alpha, r> \parallel <\beta, s>, [J_1]_\equiv\,) \in A_g^{E_1} \; |\!\} \\
v_0^{F_1} = [\overline{F_1}]_\equiv
\end{array}$$

Note that in V^{F_1}:

- $[\overline{F_1}]_\equiv = [\overline{E_1}\,[f]\,]_\equiv$

- $[\underline{F_1}]_\equiv = [\underline{E_1}\,[f]\,]_\equiv$

$$\begin{array}{l}
nts(F_2) = (V^{F_2} \cup \{[\underline{F_2}]_\equiv\}, A^{F_2} \cup A_{sr}^{F_2} \cup A_g^{F_2}, v_0^{F_2}), \quad \text{where} \quad ts(F_2) = (V^{F_2}, A^{F_2}, v_0^{F_2}) \\
V^{F_2} = \{[H_2\,[f]]_\equiv \quad | \quad H_2 \in [\overline{E_2}\,\rangle\,\} \\
A^{F_2} = \{\!| \; ([H_2\,[f]\,]_\equiv\,, <f(\alpha), r>, [J_2\,[f]\,]_\equiv\,) \,|\, ([H_2]_\equiv\,, <\alpha, r>, [J_2]_\equiv\,) \in A^{E_2} \; |\!\} \\
A_{sr}^{F_2} = \{([\overline{F_2}]_\equiv, <skip, 0>, [\underline{F_2}]_\equiv), ([\underline{F_2}]_\equiv, <redo, \infty>, [\overline{F_2}]_\equiv)\} \\
A_g^{F_2} = \{\!| \; ([H_2\,[f]\,]_\equiv\,, <f(\alpha), r> \parallel <f(\beta), s>, [J_2\,[f]\,]_\equiv\,) \,|\, ([H_2]_\equiv\,, <\alpha, r> \parallel <\beta, s>, [J_2]_\equiv\,) \in A_g^{E_2} \; |\!\} \\
v_0^{F_2} = [\overline{F_2}]_\equiv
\end{array}$$

With $[\overline{F_2}]_\equiv = [\overline{E_2}\,[f]\,]_\equiv$, and $[\underline{F_2}]_\equiv = [\underline{E_2}\,[f]\,]_\equiv$.

We define $\varphi : V^{F_1} \cup \{[\underline{F_1}]_\equiv\} \to V^{F_2} \cup \{[\underline{F_2}]_\equiv\}$ in the following way:

- $\forall [H_1]_\equiv \in V^{E_1}$, $\varphi([H_1\,[f]\,]_\equiv) = [H_2\,[f]\,]_\equiv$, where $\phi([H_1]_\equiv) = [H_2]_\equiv$.

- $\varphi([\underline{F_1}]_\equiv) = [\underline{F_2}]_\equiv$

It is immediate to check that φ is well defined, it is a bijection, and $\varphi([\overline{F_1}]_\equiv) = [\overline{F_2}]_\equiv$.

- Let us consider an edge of A^{F_1}, $([H_1\,[f]\,]_\equiv\,, <f(\alpha), r>, [J_1\,[f]\,]_\equiv\,)$, where: $([H_1]_\equiv\,, <\alpha, r>, [J_1]_\equiv\,) \in A^{E_1}$. We have:
  $r([H_1\,[f]\,]_\equiv\,, f(\alpha), [J_1\,[f]\,]_\equiv\,) = r([H_1]_\equiv\,, \alpha, [J_1]_\equiv\,) = r([H_2]_\equiv\,, \alpha, [J_2]_\equiv\,) =$
  $r([H_2\,[f]\,]_\equiv\,, f(\alpha), [J_2\,[f]\,]_\equiv\,)$, where $\phi([H_1]_\equiv) = [H_2]_\equiv$, and $\phi([J_1]_\equiv) = [J_2]_\equiv$.

- Edges of $A_g^{F_1}$ must have the following form:

  $$([H_1\,[f]\,]_\equiv\,, <f(\alpha), r> \parallel <f(\beta), s>, [J_1\,[f]\,]_\equiv\,)$$

  where $([H_1]_\equiv\,, <\alpha, r> \parallel <\beta, s>, [J_1]_\equiv\,) \in A_g^{E_1}$.

  Then, using $E_1 \sim E_2$, and taking into account that f is bijective we can easily conclude the property for $E_1[f]$ and $E_2[f]$.

(v) Let $F_1 = E_1\, sy\, a$ and $F_2 = E_2\, sy\, a$ be, with $a \in \mathcal{A}$. We have:

$$\begin{aligned}
& nts(\overline{F_1}) = (V^{F_1} \cup \{[\underline{F_1}]_\equiv\}, A^{F_1} \cup A_{sr}^{F_1} \cup A_g^{F_1}, v_0^{F_1}), \quad \text{where} \quad ts(\overline{F_1}) = (V^{F_1}, A^{F_1}, v_0^{F_1}) \\
& V^{F_1} = \{[H_1\, sy\, a]_\equiv \;\mid\; H_1 \in [\overline{E_1}\rangle\} \\
& A^{F_1} = \{\!\!\{\ ([H_1\, sy\, a]_\equiv, <\alpha,r>, [J_1\, sy\, a]_\equiv) \mid ([H_1]_\equiv, <\alpha,r>, [J_1]_\equiv) \in A^{E_1} \ \}\!\!\} + \\
& \qquad \{\!\!\{\ ([H_1\, sy\, a]_\equiv, <\alpha_1 \oplus_a \alpha_2, R_{12}>, [J_1\, sy\, a]_\equiv) \mid a \in A(\alpha_1), \hat{a} \in A(\alpha_2), \\
& \qquad \{<\alpha_1,r_1>, <\alpha_2,r_2>\} \in BC(H_1\, sy\, a) \text{ with} \\
& \qquad H_1\, sy\, a \xrightarrow{<\alpha_1,r_1>} G_1\, sy\, a \equiv G_1\, sy\, a^* \xrightarrow{<\alpha_2,r_2>} J_1\, sy\, a\ \}\!\!\} \\
& A_{sr}^{F_1} = \{([\overline{F_1}]_\equiv, <skip,0>, [\underline{F_1}]_\equiv), ([\underline{F_1}]_\equiv, <redo,\infty>, [\overline{F_1}]_\equiv)\} \\
& A_g^{F_1} = \{\!\!\{\ ([H_1\, sy\, a]_\equiv, <\alpha,r> \parallel <\beta,s>, [J_1\, sy\, a]_\equiv) \mid ([H_1]_\equiv, <\alpha,r> \parallel <\beta,s>, [J_1]_\equiv) \in A_g^{E_1} \ \}\!\!\} + \\
& \qquad \{\!\!\{\ ([H_1\, sy\, a]_\equiv, <\alpha_1 \oplus_a \alpha_2, R_{12}> \parallel <\beta,s>, [J_1\, sy\, a]_\equiv) \mid a \in A(\alpha_1), \hat{a} \in A(\alpha_2), \\
& \qquad \{<\alpha_1,r_1>, <\alpha_2,r_2>, <\beta,s>\} \in BC(H_1\, sy\, a) \text{ with} \\
& \qquad H_1\, sy\, a \xrightarrow{<\alpha_1,r_1>} G_1\, sy\, a \equiv G_1\, sy\, a^* \xrightarrow{<\alpha_2,r_2>} G_2\, sy\, a \equiv G_2\, sy\, a^* \xrightarrow{<\beta,s>} J_1\, sy\, a\ \}\!\!\}
\end{aligned}$$

where:

$$R_{12} = \frac{r_1}{cr(H_1\, sy\, a,\ <\alpha_1,r_1>)} \cdot \frac{r_2}{cr(H_1\, sy\, a,\ <\alpha_2,r_2>)} \cdot \min_{i=1,2} \{cr(H_1\, sy\, a, <\alpha_i,r_i>)\}$$

Note that in V^{F_1}:

- $[\overline{F_1}]_\equiv = [\overline{E_1}\, sy\, a]_\equiv$
- $[\underline{F_1}]_\equiv = [\underline{E_1}\, sy\, a]_\equiv$

$$\begin{aligned}
& nts(\overline{F_2}) = (V^{F_2} \cup \{[\underline{F_2}]_\equiv\}, A^{F_2} \cup A_{sr}^{F_2} \cup A_g^{F_2}, v_0^{F_2}), \quad \text{where} \quad ts(\overline{F_2}) = (V^{F_2}, A^{F_2}, v_0^{F_2}) \\
& V^{F_2} = \{[H_2\, sy\, a]_\equiv \mid H_2 \in [\overline{E_2}\rangle\} \\
& A^{F_2} = \{\!\!\{\ ([H_2\, sy\, a]_\equiv, <\alpha,r>, [J_2\, sy\, a]_\equiv) \mid ([H_2]_\equiv, <\alpha,r>, [J_2]_\equiv) \in A^{E_2} \ \}\!\!\} + \\
& \qquad \{\!\!\{\ ([H_2\, sy\, a]_\equiv, <\alpha_1 \oplus_a \alpha_2, R_{12}>, [J_2\, sy\, a]_\equiv) \mid a \in A(\alpha_1), \hat{a} \in A(\alpha_2), \\
& \qquad \{<\alpha_1,r_1>, <\alpha_2,r_2>\} \in BC(H_2\, sy\, a) \text{ with} \\
& \qquad H_2\, sy\, a \xrightarrow{<\alpha_1,r_1>} G_1'\, sy\, a \equiv G_1'\, sy\, a^* \xrightarrow{<\alpha_2,r_2>} J_2\, sy\, a\ \}\!\!\} \\
& A_{sr}^{F_2} = \{([\overline{F_2}]_\equiv, <skip,0>, [\underline{F_2}]_\equiv), ([\underline{F_2}]_\equiv, <redo,\infty>, [\overline{F_2}]_\equiv)\} \\
& A_g^{F_2} = \{\!\!\{\ ([H_2\, sy\, a]_\equiv, <\alpha,r> \parallel <\beta,s>, [J_2\, sy\, a]_\equiv) \mid ([H_2]_\equiv, <\alpha,r> \parallel <\beta,s>, [J_2]_\equiv) \in A_g^{E_2} \ \}\!\!\} + \\
& \qquad \{\!\!\{\ ([H_2\, sy\, a]_\equiv, <\alpha_1 \oplus_a \alpha_2, R_{12}> \parallel <\beta,s>, [J_2\, sy\, a]_\equiv) \mid a \in A(\alpha_1), \hat{a} \in A(\alpha_2), \\
& \qquad \{<\alpha_1,r_1>, <\alpha_2,r_2>, <\beta,s>\} \in BC(H_2\, sy\, a) \text{ with} \\
& \qquad H_2\, sy\, a \xrightarrow{<\alpha_1,r_1>} G_1'\, sy\, a \equiv G_1'\, sy\, a^* \xrightarrow{<\alpha_2,r_2>} G_2'\, sy\, a \equiv G_2'\, sy\, a^* \xrightarrow{<\beta,s>} J_2\, sy\, a\ \}\!\!\}
\end{aligned}$$

where:

$$R_{12} = \frac{r_1}{cr(H_2\, sy\, a,\ <\alpha_1,r_1>)} \cdot \frac{r_2}{cr(H_2\, sy\, a,\ <\alpha_2,r_2>)} \cdot \min_{i=1,2} \{cr(H_2\, sy\, a,\ <\alpha_i,r_i>)\}$$

With $[\overline{F_2}]_\equiv = [\overline{E_2}\, sy\, a]_\equiv$, and $[\underline{F_2}]_\equiv = [\underline{E_2}\, sy\, a]_\equiv$.

We define $\varphi : V^{F_1} \cup \{[\underline{F_1}]_\equiv\} \to V^{F_2} \cup \{[\underline{F_2}]_\equiv\}$, in the following way:

- $\forall [H_1]_\equiv \in V^{E_1}, \varphi([H_1\, sy\, a]_\equiv) = [H_2\, sy\, a]_\equiv$, where $\phi([H_1]_\equiv) = [H_2]_\equiv$.
- $\varphi([\underline{F_1}]_\equiv) = [\underline{F_2}]_\equiv$.

Note that φ is well defined, it is a bijection and $\varphi([\overline{F_1}]_\equiv) = [\overline{F_2}]_\equiv$.

- Let us consider an edge e of A^{F_1}. We have again two possible cases, the first one corresponds to one stochastic multiaction executable from $[H_1]_\equiv$, and the second one corresponds to a stochastic multiaction obtained by synchronization. Nevertheless, notice that as a consequence of the syntactical restriction that we have imposed, we cannot obtain two edges connecting the same pair of nodes, one obtained according to the first case, and the other one according to the second one. Then, we can see each case separately:

  - $e = ([H_1 \; sy \; a\,]_\equiv\,, <\alpha, r>, [J_1 \; sy \; a\,]_\equiv)$, with $([H_1]_\equiv\,, <\alpha, r>, [J_1]_\equiv) \in A^{E_1}$.
    Taking $\phi([H_1]_\equiv) = [H_2]_\equiv$ and $\phi([J_1]_\equiv) = [J_2]_\equiv$, we have:

    $$\begin{array}{l} r([H_1 \; sy \; a\,]_\equiv\,, \alpha, [J_1 \; sy \; a\,]_\equiv\,) = r([H_1]_\equiv\,, \alpha, [J_1]_\equiv) = \\ r([H_2]_\equiv\,, \alpha, [J_2]_\equiv) = r([H_2 \; sy \; a\,]_\equiv\,, \alpha, [J_2 \; sy \; a\,]_\equiv) \end{array}$$

  - $e = ([H_1 \; sy \; a\,]_\equiv\,, <\alpha_1 \oplus_a \alpha_2, R_{12}>, [J_1 \; sy \; a\,]_\equiv)$, with $a \in A(\alpha_1), \hat{a} \in A(\alpha_2)$, $\{<\alpha_1, r_1>, <\alpha_2, s_1>\} \in BC(H_1 \; sy \; a\,)$ with
    $H_1 \; sy \; a \stackrel{<\alpha_1,r_1>}{\longrightarrow} G_1 \; sy \; a \equiv G_1 \; sy \; a^* \stackrel{<\alpha_2,s_1>}{\longrightarrow} J_1 \; sy \; a$
    We apply induction on the number of times that we have applied the rule *Sy2*:
    Base case: *Sy2* has been applied once. In this case: $\{<\alpha_1, r_1>, <\alpha_2, s_1>\} \in BC(H_1)$, and we will have:

    $$\begin{array}{l} \mathit{Conflict}(H_1, <\alpha_1, r_1>) = \mathit{Conflict}(H_1 \; sy \; a\,, <\alpha_1, r_1>) = \{<\alpha_1, r_1>, \ldots, <\alpha_1, r_n>\} \\ \mathit{Conflict}(H_1, <\alpha_2, s_1>) = \mathit{Conflict}(H_1 \; sy \; a\,, <\alpha_2, s_1>) = \{<\alpha_2, s_1>, \ldots, <\alpha_2, s_m>\} \end{array}$$

    Therefore, we have the following transitions:

    $$\begin{array}{l} ([H_1]_\equiv, <\alpha_1, r_1> \parallel <\alpha_2, s_1>, [J_1]_\equiv) \in A_g^{E_1} \\ ([H_1]_\equiv, <\alpha_1, r_i>, [G_1]_\equiv) \in A^{E_1},\; ([H_1 \; sy \; a\,]_\equiv, <\alpha_1, r_i>, [G_1 \; sy \; a\,]_\equiv) \in A^{F_1},\;\; i = 1, \ldots, n_1 \\ ([H_1\,]_\equiv, <\alpha_2, s_j>, [G'_1\,]_\equiv) \in A^{E_1},\; ([H_1 \; sy \; a\,]_\equiv, <\alpha_2, s_j>, [G'_1 \; sy \; a\,]_\equiv) \in A^{F_1},\;\; j = 1, \ldots, m_2 \\ ([G_1\,]_\equiv, <\alpha_2, s_j>, [J_1\,]_\equiv) \in A^{E_1},\; ([G_1 \; sy \; a\,]_\equiv, <\alpha_2, s_j>, [J_1 \; sy \; a\,]_\equiv) \in A^{F_1},\;\; j = 1, \ldots, m_2 \\ ([G'_1\,]_\equiv, <\alpha_1, r_i>, [J_1\,]_\equiv) \in A^{E_1},\; ([G'_1 \; sy \; a\,]_\equiv, <\alpha_1, r_i>, [J_1 \; sy \; a\,]_\equiv) \in A^{F_1},\;\; i = 1, \ldots, n_1 \end{array}$$

    with $n \geq n_1$ and $m \geq m_2$. Applying Lemma 3 we may generate $n_1 \cdot m_2$ possible synchronizations, taking each pair $(<\alpha_1, r_i>, <\alpha_2, s_j>)$, for $i = 1, \ldots, n_1$, $j = 1, \ldots, m_2$. Then, for each one of these pairs $(<\alpha_1, r_i>, <\alpha_2, s_j>)$ we obtain a transition $([H_1 \; sy \; a]_\equiv, <\alpha_1 \oplus_a \alpha_2, R_{ij}>, [J_1 \; sy \; a]_\equiv) \in A^{F_1}$, where R_{ij} can be written as follows (using Lemma 4):

    $$R_{ij} = \frac{r_i}{cr1} \cdot \frac{s_j}{cr2} \cdot \min\{cr1\,, cr2\}$$

    where $cr1 = cr(H_1 \; sy \; a, <\alpha_1, r_1>)$, and $cr2 = cr(H_1 \; sy \; a, <\alpha_2, s_1>)$.
    Consequently:

    $$r([H_1 \; sy \; a\,]_\equiv, \alpha_1 \oplus_a \alpha_2,\, [J_1 \; sy \; a]_\equiv) = \sum_{\substack{i=1\ldots n_1 \\ j=1\ldots m_2}} R_{ij} = \frac{\widetilde{r}}{cr1} \cdot \frac{\widetilde{s}}{cr2} \cdot \min\{cr1,\, cr2\}$$

where $\widetilde{r} = \sum_{i=1...n_1} r_i\,,\ \widetilde{s} = \sum_{j=1...m_2} s_j\,.$

In $ts(\overline{E_2})$, using Def. 11 we conclude that there exists $H_2 \in \mathit{OpReDynExpr}$, $\phi([H_1]_\equiv) = [H_2]_\equiv$, and that we have the following edges:

$$
\begin{array}{l}
([H_2]_\equiv, <\alpha_1, u_1> \parallel <\alpha_2, v_1>, [J_2]_\equiv) \in A_g^{E_2} \\
([H_2]_\equiv, <\alpha_1, u_i>, [G_2]_\equiv) \in A^{E_2},\ ([H_2\ sy\ a\,]_\equiv, <\alpha_1, u_i>, [G_2\ sy\ a\,]_\equiv) \in A^{F_2},\ i = 1, \ldots, p_1 \\
([H_2]_\equiv, <\alpha_2, v_j>, [G'_2]_\equiv) \in A^{E_2},\ ([H_2\ sy\ a\,]_\equiv, <\alpha_2, v_j>, [G'_2\ sy\ a\,]_\equiv) \in A^{F_2},\ j = 1, \ldots, q_2 \\
([G_2]_\equiv, <\alpha_2, v_j>, [J_2\,]_\equiv) \in A^{E_2},\ ([G_2\ sy\ a\,]_\equiv, <\alpha_2, v_j>, [J_2\ sy\ a\,]_\equiv) \in A^{F_2},\ j = 1, \ldots, q_2 \\
([G'_2]_\equiv, <\alpha_1, u_i>, [J_2]_\equiv) \in A^{E_2},\ ([G'_2\ sy\ a\,]_\equiv, <\alpha_1, u_i>, [J_2\ sy\ a\,]_\equiv) \in A^{F_2},\ i = 1, \ldots, p_1
\end{array}
$$

with $\phi([J_1]_\equiv) = [J_2]_\equiv$, $\phi([G_1]_\equiv) = [G_2]_\equiv$, $\phi([G'_1]_\equiv) = [G'_2]_\equiv$. Furthermore:

$$
\begin{array}{l}
\mathit{Conflict}(H_2, <\alpha_1, u_1>) = \mathit{Conflict}(H_2\ sy\ a\,, <\alpha_1, u_1>) = \{<\alpha_1, u_1>, \ldots, <\alpha_1, u_p>\} \\
\mathit{Conflict}(H_2, <\alpha_2, v_1>) = \mathit{Conflict}(H_2\ sy\ a\,, <\alpha_2, v_1>) = \{<\alpha_2, v_1>, \ldots, <\alpha_2, v_q>\}
\end{array}
$$

with $p \geq p_1$ and $q \geq q_2$. Therefore, using again Lemma 3 and Lemma 4, we have $p_1 \cdot q_2$ possible synchronizations. For each one of these pairs $(<\alpha_1, u_i>, <\alpha_2, v_j>)$, we have $([H_2\ sy\ a]_\equiv, <\alpha_1 \oplus_a \alpha_2, S_{ij}>, [J_2\ sy\ a]_\equiv)$, where:

$$
S_{ij} = \frac{u_i}{cr1'} \cdot \frac{v_j}{cr2'} \cdot \min\{cr1',\ cr2'\}
$$

where:

$$
\begin{array}{l}
cr1' = cr(H_2\ sy\ a, <\alpha_1, u_1>) \\
cr2' = cr(H_2\ sy\ a, <\alpha_2, v_1>)
\end{array}
$$

From Lemma 4 and Lemma 5 we have now that: $cr1 = cr1'$ and $cr2 = cr2'$.

Finally, from $E_1 \sim E_2$, we have: $\widetilde{r} = \sum_{i=1...p_1} u_i$ and $\widetilde{s} = \sum_{j=1...p_2} v_j$, and thus:

$$
r([H_2\ sy\ a\,]_\equiv, \alpha_1 \oplus_a \alpha_2,\ [J_2\ sy\ a]_\equiv) = \sum_{\substack{i=1...p_1 \\ j=1...q_2}} S_{ij} = \frac{\widetilde{r}}{cr1'} \cdot \frac{\widetilde{s}}{cr2'} \cdot \min\{cr1',\ cr2'\} =
$$

$$
r([H_1\ sy\ a\,]_\equiv, \alpha_1 \oplus_a \alpha_2,\ [J_1\ sy\ a]_\equiv)
$$

General case: We assume now that e has been obtained applying n times the rule *Sy2*, i.e, the stochastic multiaction is obtained as follows:

$$
(<\beta_1, r_1> \oplus_a \ldots \oplus_a <\beta_k, r_k>) \oplus_a (<\beta_{k+1}, r_{k+1}> \oplus_a \ldots \oplus_a <\beta_{n+1}, r_{n+1}>)
$$

Therefore, we may apply the induction hypothesis for both parts, and then, with a similar reasoning to that used in the base case we may conclude the property.

On the other hand, if $r([H_1\ sy\ a\,]_\equiv, \alpha, [J_1\ sy\ a\,]_\equiv) = 0$, we can repeat a similar reasoning in order to conclude that $r([H_2\ sy\ a\,]_\equiv, \alpha, [J_2\ sy\ a\,]_\equiv\,) = 0$, where $\phi([H_1]_\equiv) = [H_2]_\equiv$, $\phi([J_1]_\equiv) = [J_2]_\equiv$.

- Let us consider an edge e of $A_g^{F_1}$. We have again two cases:

  - $e = ([H_1\, sy\, a]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1\, sy\, a]_\equiv)$, with $([H_1]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1]_\equiv) \in A^{E_1}$.

    It is immediate, taking into account that $E_1 \sim E_2$.
  - $e = ([H_1\, sy\, a]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1\, sy\, a]_\equiv)$, where: $<\alpha, r>$ has been obtained by synchronizing the stochastic multiactions $\{<\alpha_i, r_i>\}_{i=1}^n$, and $<\beta, s>$ has been obtained by synchronizing the stochastic multiactions $\{<\beta_j, s_j>\}_{j=1}^m$, with $n + m > 1$. Therefore:

    $$\{<\alpha_1, r_1>, \ldots, <\alpha_n, r_n>, <\beta_1, s_1>, \ldots, <\beta_m, s_m>\} \in BC(H_1')$$

    where $H_1' \equiv H_1$, and thus, from Prop. 2 we can obtain:

    $$\{<\alpha_1, r_1'>, \ldots, <\alpha_n, r_n'>, <\beta_1, s_1'>, \ldots, <\beta_m, s_m'>\} \in BC(H_2')$$

    for some $H_2' \in \phi([H_1]_\equiv)$. Taking now $<\alpha, r'>$ the stochastic multiaction that we can obtain by synchronizing all the stochastic multiactions in $\{<\alpha_i, r_i'>\}_{i=1}^n$, and $<\beta, s'>$ that one obtained by synchronizing $\{<\beta_j, s_j'>\}_{j=1}^m$, we have:

    $$\{<\alpha, r'>, <\beta, s'>\} \in BC(H_2'\, sy\, a)$$

    Therefore, there is a ghost transition:

    $$([H_2'\, sy\, a]_\equiv, <\alpha, r'> \parallel <\beta, s'>, [J_2\, sy\, a]_\equiv)$$

    which fulfills conditions of Def. 11, with $J_2 \in \phi([J_1]_\equiv)$.

(vi) Let $F_1 = E_1\, rs\, a$ and $F_2 = E_2\, rs\, a$ be, with $a \in \mathcal{A}$. Then:

$$nts(\overline{F_1}) = (V^{F_1} \cup \{[\underline{F_1}]_\equiv\}, A^{F_1} \cup A_{sr}^{F_1} \cup A_g^{F_1}, v_0^{F_1}), \quad \text{where} \quad ts(\overline{F_1}) = (V^{F_1}, A^{F_1}, v_0^{F_1})$$

$$\begin{aligned}
V^{F_1} &= \{[\overline{F_1}]_\equiv\} \cup \{[H_1\, rs\, a]_\equiv \mid H_1 \in [\overline{E_1}\rangle \text{ and } \exists H \equiv \overline{E_1} \text{ such that } H \stackrel{<\alpha_1, r_1>}{\longrightarrow} \ldots \stackrel{<\alpha_n, r_n>}{\longrightarrow} H_1 \\
&\quad \text{with } a, \hat{a} \notin A(\alpha_i),\ i = 1, \ldots, n\} \\
A^{F_1} &= \{\!\mid ([H_1\, rs\, a]_\equiv, <\alpha, r>, [J_1\, rs\, a]_\equiv) \mid ([H_1]_\equiv, <\alpha, r>, [J_1]_\equiv) \in A^{E_1}, \\
&\quad [H_1\, rs\, a]_\equiv, [J_1\, rs\, a]_\equiv \in V^{F_1}, \text{ and } a, \hat{a} \notin A(\alpha) \mid\!\} \\
A_{sr}^{F_1} &= \{([\overline{F_1}]_\equiv, <skip, 0>, [\underline{F_1}]_\equiv), ([\underline{F_1}]_\equiv, <redo, \infty>, [\overline{F_1}]_\equiv)\} \\
A_g^{F_1} &= \{\!\mid ([H_1\, rs\, a]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1\, rs\, a]_\equiv) \mid ([H_1]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1]_\equiv) \in A_g^{E_1} \\
&\quad [H_1\, rs\, a]_\equiv, [J_1\, rs\, a]_\equiv \in V^{F_1}, \text{ and } a, \hat{a} \notin A(\alpha) \cup A(\beta) \mid\!\} \\
v_0^{F_1} &= [\overline{F_1}]_\equiv
\end{aligned}$$

Note that in V^{F_1}:

- $[\overline{F_1}]_\equiv = [\overline{E_1}\, rs\, a]_\equiv$
- $[\underline{F_1}]_\equiv = [\underline{E_1}\, rs\, a]_\equiv$

$nts(\overline{F_2}) = (V^{F_2} \cup \{[\underline{F_2}]_\equiv\}, A^{F_2} \cup A^{F_2}_{sr} \cup A^{F_2}_g, v_0^{F_2}), \quad$ where $\quad ts(\overline{F_2}) = (V^{F_2}, A^{F_2}, v_0^{F_2})$

$V^{F_2} = \{[\overline{F_2}]_\equiv\} \cup \{[H_2\ rs\ a]_\equiv \mid H_2 \in [\overline{E_2}\rangle$ and $\exists H \equiv \overline{E_2}$ such that $H \overset{<\alpha_1, r_1>}{\longrightarrow} \dots \overset{<\alpha_n, r_n>}{\longrightarrow} H_2$ with $a, \hat{a} \notin A(\alpha_i)$, $i = 1, \dots, n\}$

$A^{F_2} = \{\!| \ ([H_2\ rs\ a]_\equiv, <\alpha, r>, [J_2\ rs\ a]_\equiv) \mid ([H_2]_\equiv, <\alpha, r>, [J_2]_\equiv) \in A^{E_2},$ $[H_2\ rs\ a]_\equiv, [J_2\ rs\ a]_\equiv \in V^{F_2}$ and $a, \hat{a} \notin A(\alpha) \ |\!\}$

$A^{F_2}_{sr} = \{([\overline{F_2}]_\equiv, <skip, 0>, [\underline{F_2}]_\equiv), ([\underline{F_2}]_\equiv, <redo, \infty>, [\overline{F_2}]_\equiv)\}$

$A^{F_2}_g = \{\!| \ ([H_2\ rs\ a]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_2\ rs\ a]_\equiv) \mid ([H_2]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_2]_\equiv) \in A^{E_2}_g,$ $[H_2\ rs\ a]_\equiv, [J_2\ rs\ a]_\equiv \in V^{F_2}$ and $a, \hat{a} \notin A(\alpha) \cup A(\beta) \ |\!\}$

$v_0^{F_2} = [\overline{F_2}]_\equiv$

With $[\overline{F_2}]_\equiv = [\overline{E_2}\ rs\ a]_\equiv$ and $[\underline{F_2}]_\equiv = [\underline{E_2}\ rs\ a]_\equiv$.

We define $\varphi : V^{F_1} \cup \{[\underline{F_1}]_\equiv\} \to V^{F_2} \cup \{[\underline{F_2}]_\equiv\}$, in the following way:

- $\forall [H_1\ rs\ a]_\equiv \in V^{F_1}$, $\varphi([H_1\ rs\ a]_\equiv) = [H_2\ rs\ a]_\equiv$, with $\phi([H_1]_\equiv) = [H_2]_\equiv$.
- $\varphi([\underline{F_1}]_\equiv = [\underline{F_2}]_\equiv$

Notice that φ is well defined, because $[H_2\ rs\ a]_\equiv \in V^{F_2}$ (we can execute in E_2 the same chain of multiactions as in E_1, possibly with different rates), φ is a bijection and $\varphi([\overline{F_1}]_\equiv) = [\overline{F_2}]_\equiv$.

- Let us consider an edge e of A^{F_1}:

$$e = ([H_1\ rs\ a]_\equiv, <\alpha, r>, [J_1\ rs\ a]_\equiv)$$

with $([H_1]_\equiv, <\alpha, r>, [J_1]_\equiv) \in A^{E_1}$, $a, \hat{a} \notin A(\alpha)$.

Then:

$$r([H_1\ rs\ a]_\equiv, \alpha, [J_1\ rs\ a]_\equiv) = r([H_1]_\equiv, \alpha, [J_1]_\equiv) =$$
$$r([H_2]_\equiv, \alpha, [J_2]_\equiv) = r([H_2\ rs\ a]_\equiv, \alpha, [J_2\ rs\ a]_\equiv)$$

with $\phi([H_1]_\equiv) = [H_2]_\equiv$, $\phi([J_1]_\equiv) = [J_2]_\equiv$.

It is also immediate that:

$r([H_1\ rs\ a]_\equiv, \alpha, [J_1\ rs\ a]_\equiv) = 0$ if and only if $r([H_2\ rs\ a]_\equiv, \alpha, [J_2\ rs\ a]_\equiv) = 0$.

- Let us consider an edge e of $A^{F_1}_g$:

$$e = ([H_1\ rs\ a]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1\ rs\ a]_\equiv)$$

with $([H_1]_\equiv, <\alpha, r> \parallel <\beta, s>, [J_1]_\equiv) \in A^{E_1}_g, a, \hat{a} \notin A(\alpha) \cup A(\beta)$.

From $E_1 \sim E_2$ we have that $\exists r', s'$ such that $([H_2]_\equiv, <\alpha, r'> \parallel <\beta, s'>, [J_2]_\equiv) \in A^{E_2}_g$, with $\phi([H_1]_\equiv) = [H_2]_\equiv$, and $\phi([J_1]_\equiv) = [J_2]_\equiv$, fulfilling the conditions of Def. 11.

Therefore:

$([H_2\ rs\ a]_\equiv, <\alpha, r'> \parallel <\beta, s'>, [J_2\ rs\ a]_\equiv) \in A^{F_2}_g$, fulfilling the conditions of Def. 11.

□

Proposition 3 For all E, F regular static s-expressions such that $E \sim F$, their corresponding CTMCs are isomorphic.

Proof: Immediate, just taking $ts(\overline{E})$ and $ts(\overline{F})$. $\square$

Proposition 4 For all E, F regular static s-expressions: $E \cong F \;\Rightarrow\; E \sim F$.

Proof: Immediate. $\square$

Consequently, we have the same equivalences that we had in PBC with respect to the isomorphism there defined.

Corollary 4 For all E, F, E' regular static s-expressions, and for all $a, b \in \mathcal{A}$, then:

(i) $E \,\|\, F \sim F \,\|\, E$

(ii) $E \,\|\, (\, F \,\|\, E' \,) \sim (\, E \,\|\, F \,) \,\|\, E'$

(iii) $E \,\Box\, F \sim F \,\Box\, E$

(iv) $E \,\Box\, (\, F \,\Box\, E' \,) \sim (\, E \,\Box\, F \,) \,\Box\, E'$

(v) $E \,;\, (\, F \,;\, E' \,) \sim (\, E \,;\, F \,) \,;\, E'$

(vi) $E \; sy\, a \; sy\, b \sim E \; sy\, b \; sy\, a$

(vii) $E \; sy\, a \; sy\, a \sim E \; sy\, a$

(viii) $E \; sy\, \hat{a} \sim E \; sy\, a$

(ix) $(\, E \,;\, F \,) \; sy\, a \sim (\, E \; sy\, a \,) \,;\, (\, F \; sy\, a \,)$

(x) $(\, E \,\Box\, F \,) \; sy\, a \sim (\, E \; sy\, a \,) \,\Box\, (\, F \; sy\, a \,)$

(xi) $(\, E \,\|\, F \,) \; sy\, a \sim (\, (\, E \; sy\, a \,) \,\|\, (\, F \; sy\, a \,) \,) sy\, a$

(xii) $E \; rs\, a \; rs\, b \sim E \; rs\, b \; rs\, a$

(xiii) $E \; rs\, a \; rs\, a \sim E \; rs\, a$

(xiv) $E \; rs\, \hat{a} \sim E \; rs\, a$

(xv) $(\, E \,;\, F \,) rs\, a \sim (\, E \, rs\, a \,) \,;\, (\, F \; rs\, a \,)$

(xvi) $(\, E \,\Box\, F \,) \; rs\, a \sim (\, E \; rs\, a \,) \,\Box\, (\, F \; rs\, a \,)$

(xvii) $(\, E \,\|\, F \,) \; rs\, a \sim (\, E \; rs\, a \,) \,\|\, (\, F \; rs\, a \,)$

(xviii) If $a \notin \{b, \hat{b}\} \Rightarrow (\, E \; sy\, a \,) \; rs\, b \sim (E \; rs\, b \,) \; sy\, a$

(xix) $(\, E \, rs\, a \,) [f] \sim (\, E \,[f] \,) \, rs\, f(a)$

(xx) $[\, a \,:\, [\, b \,:\, E \,] \,] \sim [\, b \,:\, [\, a \,:\, E \,] \,]$

(xxi) $[\, a \,:\, [\, a \,:\, E \,]\,] \sim [\, a \,:\, E \,]$

(xxii) $[\, \hat{a} \,:\, E \,] \sim [\, a \,:\, E \,]$

(xxiii) $[\, a \,:\, (\, E \,;\, F \,)\,] \sim [\, a \,:\, E \,]\,;\,[\, a \,:\, F \,]$

(xxiv) $[\, a \,:\, (\, E \,\Box\, F \,)\,] \sim [\, a \,:\, E \,] \,\Box\, [\, a \,:\, F \,]$

(xxv) Si $a \notin \{b, \hat{b}\} \Rightarrow [\, a \,:\, E \,]\, rs\; b \sim [\, a \,:\, (\, E\, rs\; b \,)\,]$

(xxvi) If f is bijection, then $[a \,:\, E][f] = [f(a) \,:\, E[f]\,]$

(xxvii) Si $a \notin \{b, \hat{b}\} \Rightarrow [\, a \,:\, E \,]\, sy\; b \sim [\, a \,:\, (\, E\, sy\; b \,)\,]$

(xxviii) $E\,[f][g] \sim E\,[g \circ f]$

(xxix) $E\,[id] \sim E$

(xxx) $(\, E \,;\, F \,)[f] \sim (\, E\,[f]\,)\,;\,(\, F\,[f]\,)$

(xxxi) $(\, E \,\|\, F \,)[f] \sim (\, E\,[f]\,) \,\|\, (\, F\,[f]\,)$

(xxxii) $(\, E \,\Box\, F \,)[f] \sim (\, E\,[f]\,) \,\Box\, (\, F\,[f]\,)$

(xxxiii) $(\, E\, sy\; a \,)\,[f] \sim E\,[f]\, sy\; (f(a))$

$\Box$

4 Conclusions and Future Work

sPBC is a stochastic extension of PBC, which was presented in [12]. An important difference with respect to PBC is that in sPBC a total order semantics is considered, although parallelism is maintained at the level of multiactions. In this paper we have defined a congruence relation for finite sPBC ($\sim$), with which we may identify those processes that have the same behaviour, not only in terms of the multiactions that they can perform, but also for the stochastic information that they have associated. In order to do that, a new version for the semantics of the synchronization has been considered, on the basis of *conflict rates*. Furthermore, for each regular static s-expression E we have defined a new labelled transition system, $nts(\overline{E})$, in which some new transitions have been added, namely, *Skp, Rdo*, and the *ghosts transitions*.

Our future work will focus on the definition of a more general stochastic equivalence relation: a stochastic bisimulation. We also intend to extend our results to the infinite case, by including both iteration and recursion. Finally, we will introduce additional capabilities, such as immediate multiactions.

References

[1] M. Ajmone Marsan, G. Balbo, G. Conte, S. Donatelli, and G. Franceschinis. *Modelling with Generalized Stochastic Petri Nets.* Wiley, 1995.

[2] J.C.M. Baeten. *The Total Order Assumption.* In Proceedings Workshop *"What good is partial order"*, Sheffiel (E. Best, ed.), Hildesheimer Informatik-Berichte 13/92, Universitat Hildesheim 1992, pp. 1-11.

[3] E. Best, R. Devillers, and M. Koutny. *A Consistent Model for Nets and Process Algebras,* in the book *The Handbook on Process algebras, Chapter 14, pag. 873-944* (Bergstra, J.A. and Ponse, A. and Smolka, S.S.,eds), North Holland, 2001.

[4] E. Best, R. Devillers, and M. Koutny. *Petri Nets, Process Algebras and Concurrent Programming Languages,* Reisig, W. and Rozenberg, G. (eds). LNCS: Lecture on Petri Nets II: Applications, 1492, 1998.

[5] E. Best, R. Devillers, and M. Koutny. *Petri Nets, Process Algebras and Programming Languages.* Lectures on Petri Nets II: Applications, W. Reisig and Rozenberg (eds.), Advances in Petri Nets, 1-84, Springer-Verlag, 1998.

[6] E. Best, R. Devillers, and M. Koutny. *Petri Net Algebra.* EATC, Springer, 2001.

[7] E. Best, R. Devilllers, and J. Hall. *The Box Calculus : A New Causal Algebra with Multi-label Communication.* Advances in Petri Nets. Springer, LNCS, 609: 21-69, 1992.

[8] E. Best and M. Koutny. *A Refined View of the Box Algebra.* Proc. Application and Theory of Petri Nets. Springer, LNCS, 935:1-20, 1995.

[9] M. Koutny. *A Compositional Model of Time Petri Nets.* International Conference on Theory and Application of Petri Nets, 2000, Lecture Notes in Computer Science no. 1825, pp. 303-322, 2000.

[10] M. Koutny, J. Esparza, and E. Best. *Operational Semantics for the Petri Box Calculus.* Proc. of CONCUR'94 (ed. B. Jonsson and J. Parrow). Springer, LNCS, 836:210–225, 1994.

[11] H. Macià, V. Valero, F. Cuartero, and F.L. Pelayo. *A new proposal for the synchronization in sPBC.* Technical Report, DIAB-02-01-26, Department of Computer Science, University of Castilla-La Mancha, June 2002.

[12] H. Macià, V. Valero, and D. de Frutos-Escrig. *sPBC: A Markovian Extension of Finite Petri Box Calculus.* Proc. 9th Int. Workshop on Petri Nets and Performance Models (IEEE), 2001 pp. 207-216.

[13] O. Marroquín and D. de Frutos. *Extending the Petri Box calculus with Time.* Proc. Int. Conf. on Theory and Application of Petri Nets 2001. Springer, LNCS. Vol. 2075, 2001.

[14] R. Milner. *Communication and Concurrency.* Prentice-Hall International, 1989.

[15] G. D. Plotkin. *A Structural Approach to Operational Semantics.* Computer Science Department, University of Aarhus, 1981.